SEMINORMAL FORMS AND CYCLOTOMIC QUIVER HECKE ALGEBRAS OF TYPE A

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ABSTRACT. This paper shows that the cyclotomic quiver Hecke algebras of type A, and the gradings on these algebras, are intimately related to the classical seminormal forms. We start by classifying all seminormal bases and then give an explicit "integral" closed formula for the Gram determinants of the Specht modules in terms of the combinatorics which utilizes the KLR gradings. We then use seminormal forms to give a deformation of the KLR algebras of type A. This makes it possible to study the cyclotomic quiver Hecke algebras in terms of the semisimple representation theory and seminormal forms. As an application we construct a new distinguished graded cellular basis of the cyclotomic KLR algebras of type A.

CONTENTS

1. INTRODUCTION

The quiver Hecke algebras are a remarkable family of algebras which were introduced independently by Khovanov and Lauda [20, 21] and Rouquier [29]. These algebras are attached to an arbitrary oriented quiver, they are $\mathbb{Z}$-graded and they categorify the negative part of the associated quantum group. Over a field, Brundan and Kleshchev showed that the cyclotomic quiver Hecke algebras of type A, which are certain quotients of the quiver Hecke algebras of type A, are isomorphic to the cyclotomic Hecke algebras of type A.

The quiver Hecke algebras have a homogeneous presentation by generators and relations. As a consequence they have well-defined integral forms. Unlike Hecke algebras, which are generically semisimple, the cyclotomic quiver Hecke algebras are intrinsically non-semisimple algebras. This implies that the cyclotomic quiver Hecke algebras cannot be isomorphic to the cyclotomic Hecke algebras over an arbitrary ring.

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The first main result of this paper shows that the cyclotomic quiver Hecke algebras of type A admit a one-parameter deformation. Moreover, this deformation is isomorphic to cyclotomic Hecke algebra defined over the corresponding ring. Before we can state this result we need some notation.

Fix integers $n \geq 0$ and $e > 1$ and let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \to i+1$, for $i \in I$. Given $i \in I$ let $\hat{\imath} \geq 0$ be the smallest non-negative integer such that $i = \hat{\imath} + e\mathbb{Z}$. For each dominant weight Λ for the corresponding Kac-Moody algebra $\mathfrak{g}(\Gamma_e)$, there exists a cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$ and a cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$. To each tuple $\mathbf{i} \in I^n$ we associate the set of standard tableaux $\operatorname{Std}(\mathbf{i})$ with residue sequence $\mathbf{i}$ and for $1 \leq r \leq n$ we set

$$\mathscr{E}_r(\mathbf{i}) = \{\, c_r(\mathfrak{t}) - \hat{\imath}_r \mid \mathfrak{t} \in \operatorname{Std}(\mathbf{i}) \,\} \subset \mathbb{Z},$$

where $c_r(\mathfrak{t}) \in \mathbb{Z}$ is the *content* of r in the tableau $\mathfrak{t}$. These definitions ensure that $\mathscr{E}_r(\mathbf{i}) \subseteq \{\, ke \mid k \in \mathbb{Z} \,\}$. All of these terms are defined in Section 3.1.

Like the cyclotomic quiver Hecke algebra, our deformation of $\mathcal{R}_n^\Lambda$ is adapted to the choice of e through the choice of base ring $\mathcal{O}$ which must be an *e-idempotent subring* (Definition 4.1). This definition ensures that the cyclotomic Hecke algebras are semisimple over the field of fractions of $\mathcal{O}$ and that $\mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$ is a cyclotomic quiver Hecke algebra whenever $K = \mathcal{O}/\mathfrak{m}$, for $\mathfrak{m}$ a maximal ideal of $\mathcal{O}$. For $t \in \mathcal{O}$ and $k \in \mathbb{Z}$ let $[k] = [k]_t$ be the corresponding quantum integer.

We can now state our first main result.

Theorem A. *Suppose that $(\mathcal{O}, t)$ is an e-idempotent subring of a field $\mathscr{K}$. Then the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n \,\} \cup \{\, \psi_r^{\mathcal{O}} \mid 1 \leq r < n \,\} \cup \{\, y_r^{\mathcal{O}} \mid 1 \leq r \leq n \,\}$$

subject only to the following relations:

$$\prod_{c \in \mathscr{E}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0,$$

$$f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{ij}} f_{\mathbf{i}}^{\mathcal{O}}, \qquad \textstyle\sum_{\mathbf{i} \in I^n} f_{\mathbf{i}}^{\mathcal{O}} = 1, \qquad y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} y_r^{\mathcal{O}},$$

$$\begin{aligned}
\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= f_{s_r \cdot \mathbf{i}}^{\mathcal{O}} \psi_r^{\mathcal{O}}, & y_r^{\mathcal{O}} y_s^{\mathcal{O}} &= y_s^{\mathcal{O}} y_r^{\mathcal{O}}, \\
\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= (y_r^{\mathcal{O}} \psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, & y_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= (\psi_r^{\mathcal{O}} y_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, \\
\psi_r^{\mathcal{O}} y_s^{\mathcal{O}} &= y_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, & &\text{if } s \neq r, r+1, \\
\psi_r^{\mathcal{O}} \psi_s^{\mathcal{O}} &= \psi_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, & &\text{if } |r - s| > 1,
\end{aligned}$$

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftrightarrows i_{r+1}, \\
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \to i_{r+1}, \\
(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1}, \\
0, & \text{if } i_r = i_{r+1}, \\
f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,}
\end{cases}$$

$$\big(\psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}}\big) f_{\mathbf{i}}^{\mathcal{O}} =$$

$$\begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\
-t^{1+\rho_r(\mathbf{i})} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \to i_{r+1}, \\
f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\
0, & \text{otherwise,}
\end{cases}$$

where $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1}$ and $y_r^{\langle d\rangle} = t^d y_r^{\mathcal{O}} + [d]$, for $d \in \mathbb{Z}$.

Almost all of the relations in Theorem A appear in the presentation of the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$. The KLR relations of $\mathcal{R}_n^\Lambda$ which are 'deformed' are the quadratic relations for $\psi_r^{\mathcal{O}}$, the "braid relations" of length 3 for the $\psi_r^{\mathcal{O}}$ and the "cyclotomic relation" for $y_1^{\mathcal{O}}$. Interestingly, only the "Jucys-Murphy like elements" $y_r^{\mathcal{O}}$ need to be modified in order to define a deformation of $\mathcal{R}_n^\Lambda$. Over a field $K = \mathcal{O}/\mathfrak{m}$, the presentation in Theorem A collapses to give the KLR algebra $\mathcal{R}_n^\Lambda$ because the definition of an idempotent subring ensures that $t^{1+\rho_r(\mathbf{i})} \otimes 1_K = 1$ and $y_r^{\langle 1\pm\rho_r(\mathbf{i})\rangle} \otimes 1_K = y_r^{\mathcal{O}} \otimes 1_K$, for $1 \le r \le n$.

Theorem A also imposes additional "cyclotomic relations" on $y_r^{\mathcal{O}}$, for $2 \le r \le n$, which do not appear in the presentation of $\mathcal{R}_n^\Lambda$. These extra relations are probably redundant, however, we use them to show that the algebra defined by the presentation in Theorem A is finite dimensional. In this paper we show that analogues of these "extra" relations hold in $\mathcal{R}_n^\Lambda$, thus giving new upper bounds on the nilpotency index of the elements $y_1, \dots, y_n$ in the cyclotomic quiver Hecke algebras of type A.

To prove Theorem A we work almost entirely inside the semisimple representation theory of the cyclotomic Hecke algebras $\mathcal{H}_n^\Lambda$. We show that definition of the quiver Hecke algebra $\mathcal{R}_n^\Lambda$, and its grading, is implicit in *Young's seminormal form*. With hindsight, using the perspective afforded by this paper, it is not too much of an exaggeration to say that Murphy could have discovered the cyclotomic quiver Hecke algebras in 1983 soon after writing his paper on the Nakayama conjecture [27].

Our proof of Theorem A gives an explanation for the KLR relations and a more conceptual proof of one direction in Brundan and Kleshchev's graded isomorphism theorem [6]. Using our framework it is possible to give a completely new proof of the isomorphisms $\mathcal{R}_n^\Lambda(K) \cong \mathcal{H}_n^\Lambda(K)$, when K is a field, however, it is more convenient for us to use the existence of these isomorphisms to bound the dimension of the algebras defined by the presentation in Theorem A.

For the algebras of type A the authors constructed a graded cellular basis $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ for $\mathcal{R}_n^\Lambda$ [15]. Here $\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ is the set of all pairs of standard tableaux of the same shape, which is a multipartition of n. The element $\psi_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg_e \mathfrak{s} + \deg_e \mathfrak{t}$, where $\deg_e : \operatorname{Std}(\mathcal{P}_n^\Lambda) \longrightarrow \mathbb{Z}$ is the combinatorial degree function introduced by Brundan, Kleshchev and Wang [8]. Li [23] has shown that $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is a graded cellular basis of $\mathcal{R}_n^\Lambda$ over an arbitrary ring. In particular, the KLR algebra $\mathcal{R}_n^\Lambda$ is always free of rank $\dim \mathcal{H}_n^\Lambda(K)$, for K a field.

One of the problems with the basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ is that, because the KLR generators ψ_r, for $1 \le r < n$, do not satisfy the braid relations, the basis elements $\psi_{\mathfrak{s}\mathfrak{t}}$ depend upon a choice of reduced expression for the permutations $d(\mathfrak{s}), d(\mathfrak{t}) \in \mathfrak{S}_n$ (see Section 2.4). One of the consequences of Theorem A is that we obtain a new graded cellular basis for $\mathcal{H}_n^\Lambda$ which is independent of such choices.

Theorem B. *Suppose that K is a field. Then $\mathcal{H}_n^\Lambda(K)$ has a graded cellular basis*

$$\{\, B_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$$

where $\deg B_{\mathfrak{s}\mathfrak{t}} = \deg_e \mathfrak{s} + \deg_e \mathfrak{t}$, for $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$, such that $B_{\mathfrak{s}\mathfrak{t}}$ depends only on $\mathfrak{s}$ and $\mathfrak{t}$ and not on the choice of reduced expressions for the coset representatives $d(\mathfrak{s})$ and $d(\mathfrak{t})$.

For $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ the basis element $B_{\mathfrak{s}\mathfrak{t}}$ is uniquely determined in a way that is reminiscent of the Kazhdan-Lusztig basis. That is, we show that there exists a unique element $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ such that

$$B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})} p_{\mathfrak{u}\mathfrak{v}}^{\mathfrak{s}\mathfrak{t}}(x^{-1}) f_{\mathfrak{u}\mathfrak{v}},$$

where $p^{\mathfrak{st}}_{\mathfrak{uv}}(x) \in xK[x]$ and where $\{f_{\mathfrak{st}}\}$ is a seminormal basis of $\mathcal{H}^\Lambda_n$ which is adapted to the KLR setting. Moreover, if K is a field of characteristic zero then $\deg p^{\mathfrak{st}}_{\mathfrak{uv}}(x) \le \frac{1}{2}(\deg \mathfrak{u} - \deg \mathfrak{s} + \deg \mathfrak{v} - \deg \mathfrak{t})$.

To prove the two theorems above, we define a seminormal basis of a semisimple Hecke algebra to be a basis of $\mathcal{H}^\Lambda_n$ of simultaneous eigenvectors for the Gelfand-Zetlin subalgebra of $\mathcal{H}^\Lambda_n$. Seminormal bases are classical objects which are ubiquitous in the literature, having been rediscovered many times since were first introduced for the symmetric groups by Young in 1900 [34].

This paper starts by classifying all seminormal bases in terms of *seminormal coefficient systems*. As far as we are aware this is the first time a classification of seminormal bases has appeared in the literature, however, the real surprise is that seminormal coefficient systems encode the KLR grading.

The close connections between the semisimple representation theory and the KLR gradings is made even more explicit in the third main result of this paper which gives a closed formula for the Gram determinants of the semisimple Specht modules of these algebras. Closed formulas for these determinants already exist in the literature [4, 16–18], however, all of these formulas describe these determinants as rational functions (or rational numbers in the degenerate case). The theorem below gives the first *integral* formula for these determinants.

In order to state the closed integral formulas for the Gram determinant of the Specht module $S^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda}$ a multipartition, define

$$\deg_e(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \deg_e(\mathfrak{t}) \in \mathbb{Z},$$

where $\operatorname{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Let $\Phi_e(t) \in \mathbb{Z}[t]$ be the eth cyclotomic polynomial for $e > 1$. We prove the following (see Theorem 3.22 for a more precise statement).

Theorem C. *Suppose that $\mathcal{H}^\Lambda_n$ is a semisimple cyclotomic Hecke algebra over $\mathbb{Q}(t)$, with Hecke parameter t. Let $\boldsymbol{\lambda}$ be a multipartition of n. Then the Gram determinant of the Specht module $S^{\boldsymbol{\lambda}}$ is equal to*

$$t^N \prod_{e>1} \Phi_e(t)^{\deg_e(\boldsymbol{\lambda})},$$

for a known integer N. In particular, $\deg_e(\boldsymbol{\lambda}) \ge 0$, for all $e \in \{0, 2, 3, 4, \dots\}$.

As the integers $\deg_e(\boldsymbol{\lambda})$ are defined combinatorially, it should be possible to give a purely combinatorial proof that $\deg_e(\boldsymbol{\lambda}) \ge 0$. In Section 3.3 we give two representation theoretic proofs of this result. The first proof is elementary but not very informative. The second proof uses deep positivity properties of the graded decomposition numbers of $\mathcal{H}^\Lambda_n(\mathbb{C})$ to show that the tableaux combinatorics of $\mathcal{H}^\Lambda_n$ provides a framework for giving purely combinatorial formulas for the graded dimensions of the simple $\mathcal{H}^\Lambda_n$-modules and for the graded decomposition numbers of $\mathcal{H}^\Lambda_n$. Interestingly, we show that there is a close connection between the graded dimensions of the simple $\mathcal{H}^\Lambda_n$-modules and the graded decomposition numbers for $\mathcal{H}^\Lambda_n$. Note that in characteristic zero, the graded decomposition numbers of $\mathcal{H}^\Lambda_n$ are parabolic Kazhdan-Lusztig polynomials of type A [7], so our results show that the tableaux combinatorics leads to combinatorial formulas for these polynomials. Unfortunately, we are only able to prove that such formulas exist and we are not able to make them explicit or to show that they are canonical in any way.

The outline of this paper is as follows. Chapter 2 defines the cyclotomic Hecke algebras of type A, giving a uniform presentation for the degenerate and non-degenerate algebras. Previously these algebras have been treated separately in the literature. We then recall the basic results about these algebras that we need

from the literature, including Brundan and Kleshchev's graded isomorphism theorem [6]. Chapter 3 develops the theory of seminormal bases for these algebras in full generality. We completely classify the seminormal bases of $\mathcal{H}_n^\Lambda$ and then use them to prove Theorem C, thus establishing a link between the semisimple representation theory of $\mathcal{H}_n^\Lambda$ and the quiver Hecke algebra $\mathcal{R}_n^\Lambda$. Using this we prove the existence of combinatorial formulas for the graded dimensions of the simple modules and the graded decomposition numbers of $\mathcal{H}_n^\Lambda$. In Chapter 4 we use the theory of seminormal forms to construct a deformation of the cyclotomic quiver Hecke algebras of type A, culminating with the proof of Theorem A. Chapter 5 builds on Theorem A to give a quicker construction of the graded cellular basis of $\mathcal{H}_n^\Lambda(K)$, over a field K, which was one of the main results of [15]. Finally, in Chapter 6 we use Theorem A to show that $\mathcal{H}_n^\Lambda(K)$ has the distinguished graded cellular basis described in Theorem B.

2. Cyclotomic Hecke algebras

This chapter defines the cyclotomic Hecke and quiver Hecke algebras of type A and it introduces some of the basic machinery that we need for understanding these algebras. We give a new presentation for the *cyclotomic Hecke algebras of type A*, which simultaneously captures the degenerate and non-degenerate cyclotomic Hecke algebras which currently appear in the literature, and then we recall the results from the literature that we need, including Brundan and Kleshchev's graded isomorphism theorem [6].

2.1. **Quiver combinatorics.** Fix an integer $e \in \{0, 2, 3, 4 \dots\}$ and let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \longrightarrow i+1$, for $i \in I$. If $i, j \in I$ and i and j are not connected by an edge in Γ_e then we write $i \not\!\!- j$.

To the quiver Γ_e we attach the Cartan matrix $(c_{ij})_{i,j \in I}$, where

$$c_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \to j \text{ or } i \leftarrow j, \\ -2, & \text{if } i \leftrightarrows j, \\ 0, & \text{otherwise,} \end{cases}$$

Let $\widehat{\mathfrak{sl}}_e$ be the corresponding Kac-Moody algebra [19] with fundamental weights $\{\, \Lambda_i \mid i \in I \,\}$, positive weight lattice $P_e^+ = \sum_{i \in I} \mathbb{N}\Lambda_i$ and positive root lattice $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. Let $(\cdot, \cdot)$ be the bilinear form determined by

$$(\alpha_i, \alpha_j) = c_{ij} \qquad \text{and} \qquad (\Lambda_i, \alpha_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

More details can be found, for example, in [19, Chapter 1].

Fix, once and for all, a **multicharge** $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ which is a sequence of integers such that if $e \neq 0$ then $\kappa_l - \kappa_{l+1} \geq n$ for $1 \leq l < \ell$. Define $\Lambda = \Lambda_e(\boldsymbol{\kappa}) = \Lambda_{\bar\kappa_1} + \dots + \Lambda_{\bar\kappa_\ell}$, where $\bar\kappa = \kappa \pmod e$. Equivalently, Λ is the unique element of P_e^+ such that

$$(\Lambda, \alpha_i) = \#\{\, 1 \leq l \leq \ell \mid \kappa_l \equiv i \pmod e \,\}, \qquad \text{for all } i \in I. \tag{2.1}$$

All of the bases for the modules and algebras in this paper depend implicitly on the choice of $\boldsymbol{\kappa}$ even though the algebras themselves depend only on Λ.

2.2. **Cyclotomic Hecke algebras.** This section defines the cyclotomic Hecke algebras of type A and explains the connection between these algebras and the degenerate and non-degenerate Hecke algebras of type $G(\ell, 1, n)$.

Fix an integral domain $\mathcal{O}$ which contains an invertible element $\xi \in \mathcal{O}^\times$.

2.2. **Definition.** *Fix integers $n \ge 0$ and $\ell \ge 1$. Then the **cyclotomic Hecke algebra of type** A with Hecke parameter $\xi \in \mathcal{O}^\times$ and cyclotomic parameters $Q_1, \dots, Q_\ell \in \mathcal{O}$ is the unital associative $\mathcal{O}$-algebra $\mathcal{H}_n = \mathcal{H}_n(\mathcal{O}, \xi, Q_1, \dots, Q_\ell)$ with generators $L_1, \dots, L_n$, $T_1, \dots, T_{n-1}$ which are subject to the relations*

$$\begin{aligned}
&\prod_{l=1}^{\ell}(L_1 - Q_l) = 0, && (T_r+1)(T_r-\xi) = 0, && \\
&L_rL_t = L_tL_r, && T_rT_s = T_sT_r && \text{if } |r-s|>1,\\
&T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1}, && T_rL_t = L_tT_r, && \text{if } t \ne r, r+1,\\
&&L_{r+1}(T_r - \xi + 1) = T_rL_r + 1,&&
\end{aligned}$$

where $1 \le r < n$, $1 \le s < n-1$ and $1 \le t \le n$.

2.3. *Remark.* If $\xi = 1$ then, by definition, $\mathcal{H}_n$ is a degenerate cyclotomic Hecke algebra of type $G(\ell,1,n)$. If $\xi \ne 1$ then $\mathcal{H}_n$ is (isomorphic to) an integral cyclotomic Hecke algebra of type $G(\ell,1,n)$. To see this define $L'_k = (\xi-1)L_k+1$, for $1 \le k \le n$, and observe that $\mathcal{H}_n$ is generated by $L'_1, T_1, \dots, T_{n-1}$ subject to the usual relations for these algebras as originally defined by Ariki and Koike [3]. It is now easy to verify our claim. The presentation of $\mathcal{H}_n$ in Definition 2.2 unifies the definition of the 'degenerate' and 'non-degenerate' Hecke algebras, which corresponds to the cases where $\xi = 1$ or $\xi \ne 1$, respectively.

Let $\mathfrak{S}_n$ be the **symmetric group** on n letters. For $1 \le r < n$ let $s_r = (r, r+1)$ be the corresponding simple transposition. Then $\{s_1, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. A **reduced expression** for $w \in \mathfrak{S}_n$ is a word $w = s_{r_1}, \dots s_{r_k}$ with k minimal and $1 \le r_j < n$ for $1 \le j \le k$. If $w = s_{r_1} \dots s_{r_k}$ is reduced then set $T_w = T_{r_1} \dots T_{r_k}$. Then T_w is independent of the choice of reduced expression since the braid relations hold in $\mathcal{H}_n$. It follows arguing as in [3, Theorem 3.3] that $\mathcal{H}_n$ is free as a $\mathcal{O}$-module with basis

$$\{\, L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_1, \dots, a_n < \ell \text{ and } w \in \mathfrak{S}_n \,\}.$$

Consequently, $\mathcal{H}_n$ is free as a $\mathcal{O}$-module of rank $\ell^n n!$, which is the order of the complex reflection group of type $G(\ell,1,n)$.

We now restrict our attention to the case of *integral* cyclotomic parameters. To define these recall that for any integer k and $t \in \mathcal{O}$ the **quantum integer** $[k]_t$ is

$$[k]_t = \begin{cases} 1 + t + \dots + t^{k-1}, & \text{if } k \ge 0,\\ -(t^{-1} + t^{-2} + \dots + t^{k}), & \text{if } k < 0.\end{cases}$$

When t is understood we simply write $[k] = [k]_t$.

An **integral cyclotomic Hecke algebra** is a cyclotomic Hecke algebra $\mathcal{H}_n$ with cyclotomic parameters of the form $Q_r = [\kappa_r]_\xi$, for $\kappa_1, \dots, \kappa_\ell \in \mathbb{Z}$. The sequence of integers $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$. is the **multicharge** of $\mathcal{H}_n$.

Translating the Morita equivalence theorems of [11, Theorem 1.1] and [5, Theorem 5.19] into the current setting, every cyclotomic Hecke algebras of type A is Morita equivalent to a direct sum of tensor products of integral cyclotomic Hecke algebras. Therefore, there is no loss of generality in restricting our attention to the integral cyclotomic Hecke algebras of type A.

Recall that $\Lambda \in P_e^+$ and that we have fixed an integer $e \in \{0, 2, 3, 4, \dots\}$. Let $\xi \in \mathcal{O}^\times$ be a primitive eth root of unity if $e > 0$ and a non-root of unity if $e = 0$ and fix a multicharge $\boldsymbol{\kappa}$ so that $\Lambda = \Lambda_e(\boldsymbol{\kappa})$ as in (2.1).

Let $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{O})$ be the integral cyclotomic Hecke algebra $\mathcal{H}_n(\mathcal{O}, \xi, \boldsymbol{\kappa})$. Using the definitions it is easy to see that, up to isomorphism, $\mathcal{H}_n^\Lambda$ depends only on ξ and Λ. In fact, by Theorem 2.14 below, it depends only on e and Λ. Nonetheless,

many of the constructions which follow, particularly the definitions of bases, depend upon the choice of $\boldsymbol{\kappa}$.

2.3. **Graded algebras and cellular bases.** This section recalls the definitions and results from the representation theory of (graded) cellular algebras that we need.

Let A be a unital associative $\mathcal{O}$-algebra which is free and of finite rank as an $\mathcal{O}$-module. In this paper a **graded module** will always mean a $\mathbb{Z}$-graded module. That is, an $\mathcal{O}$-module M which has a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_d$ as an $\mathcal{O}$-module. If $m \in M_d$, for $d \in \mathbb{Z}$, then m is **homogeneous** of **degree** d and we set $\deg m = d$. If M is a graded $\mathcal{O}$-module and $s \in \mathbb{Z}$ let $M\langle s\rangle$ be the graded $\mathcal{O}$-module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

Similarly a **graded algebra** is a unital associative $\mathcal{O}$-algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ which is a graded $\mathcal{O}$-module such that $A_dA_e \subseteq A_{d+e}$, for all $d, e \in \mathbb{Z}$. It follows that $1 \in A_0$ and that A_0 is a graded subalgebra of A. A graded (right) A-module is a graded $\mathcal{O}$-module M such that $\underline{M}$ is an $\underline{A}$-module and $M_dA_e \subseteq M_{d+e}$, for all $d, e \in \mathbb{Z}$, where $\underline{M}$ and $\underline{A}$ mean forgetting the $\mathbb{Z}$-grading structures on M and A respectively. Graded submodules, graded left A-modules and so on are all defined in the obvious way.

The following definition extends Graham and Lehrer's [12] definition of cellular algebras to the graded setting.

2.4. **Definition** (Graded cellular algebras [12,15])**.** *Suppose that A is an $\mathcal{O}$-algebra which is free of finite rank over $\mathcal{O}$. A* ***cell datum*** *for A is an ordered triple $(\mathcal{P}, T, C)$, where $(\mathcal{P}, \rhd)$ is the* ***weight poset****, $T(\lambda)$ is a finite set for $\lambda \in \mathcal{P}$, and*

$$C\colon \coprod_{\lambda\in\mathcal{P}} T(\lambda)\times T(\lambda) \longrightarrow A; (\mathfrak{s},\mathfrak{t}) \mapsto c_{\mathfrak{s}\mathfrak{t}},$$

is an injective function such that:

(GC$_1$) *$\{\, c_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda)$ for $\lambda \in \mathcal{P}\,\}$ is an $\mathcal{O}$-basis of A.*

(GC$_2$) *If $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, for some $\lambda \in \mathcal{P}$, and $a \in A$ then there exist scalars $r_{\mathfrak{t}\mathfrak{v}}(a)$, which do not depend on $\mathfrak{s}$, such that*

$$c_{\mathfrak{s}\mathfrak{t}}a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a)c_{\mathfrak{s}\mathfrak{v}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda}$ is the $\mathcal{O}$-submodule of A spanned by $\{\, c_{\mathfrak{a}\mathfrak{b}} \mid \mu \rhd \lambda$ and $\mathfrak{a},\mathfrak{b} \in T(\mu)\,\}$.

(GC$_3$) *The $\mathcal{O}$-linear map $\star\colon A \longrightarrow A$ determined by $(c_{\mathfrak{s}\mathfrak{t}})^\star = c_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \mathcal{P}$ and all $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, is an anti-isomorphism of A.*

A ***cellular algebra*** *is an algebra which has a cell datum. If A is a cellular algebra with cell datum $(\mathcal{P}, T, C)$ then the basis $\{\, c_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathcal{P}$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda)\,\}$ is a* ***cellular basis*** *of A with $*$ its cellular algebra anti-automorphism.*

If, in addition, A is a $\mathbb{Z}$-graded algebra then a ***graded cell datum*** *for A is a cell datum $(\mathcal{P}, T, C)$ together with a degree function*

$$\deg : \coprod_{\lambda\in\mathcal{P}} T(\lambda) \longrightarrow \mathbb{Z}$$

such that

(GC$_d$) *the element $c_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg c_{\mathfrak{s}\mathfrak{t}} = \deg(\mathfrak{s}) + \deg(\mathfrak{t})$, for all $\lambda \in \mathcal{P}$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda)$.*

In this case, A is a ***graded cellular algebra*** *with* ***graded cellular basis*** *$\{c_{\mathfrak{s}\mathfrak{t}}\}$.*

Fix a (graded) cellular algebra A with graded cellular basis $\{c_{\mathfrak{s}\mathfrak{t}}\}$. If $\lambda \in \mathcal{P}$ then the graded **cell module** is the $\mathcal{O}$-module C^λ with basis $\{\, c_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\,\}$ and with

A-action

$$c_{\mathfrak{t}}a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a)c_{\mathfrak{v}},$$

where the scalars $r_{\mathfrak{t}\mathfrak{v}}(a) \in \mathcal{O}$ are the same scalars appearing in (GC$_2$). One of the key properties of the graded cell modules is that by [15, Lemma 2.7] they come equipped with a homogeneous bilinear form $\langle\ ,\ \rangle$ of degree zero which is determined by the equation

$$\langle c_{\mathfrak{t}}, c_{\mathfrak{u}}\rangle c_{\mathfrak{s}\mathfrak{v}} \equiv c_{\mathfrak{s}\mathfrak{t}}c_{\mathfrak{u}\mathfrak{v}} \pmod{A^{\rhd\lambda}}, \tag{2.5}$$

for $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in T(\lambda)$. The radical of this form

$$\operatorname{rad} C^\lambda = \{\, x \in C^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in C^\lambda \,\}$$

is a graded A-submodule of C^λ so that $D^\lambda = C^\lambda/\operatorname{rad} C^\lambda$ is a graded A-module. It is shown in [15, Theorem 2.10] that

$$\{\, D^\lambda\langle k\rangle \mid \lambda \in \mathcal{P}, D^\lambda \neq 0 \text{ and } k \in \mathbb{Z} \,\}$$

is a complete set of pairwise non-isomorphic irreducible (graded) A-modules when $\mathcal{O}$ is a field.

2.4. **Multipartitions and tableaux.** A **partition** of d is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $|\lambda| = \lambda_1 + \lambda_2 + \dots = d$. An **$\ell$-multipartition** of n is an ℓ-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = n$. We identify the multipartition $\boldsymbol{\lambda}$ with its **diagram** which is the set of **nodes** $[\![\boldsymbol{\lambda}]\!] = \{\, (l, r, c) \mid 1 \le c \le \lambda^{(l)}_r \text{ for } 1 \le l \le \ell \,\}$, which we think of as an ordered ℓ-tuple of arrays of boxes in the plane. For example, if $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$ then

$$[\![\boldsymbol{\lambda}]\!] = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\right).$$

In this way we talk of the **rows**, **columns** and **components** of $\boldsymbol{\lambda}$.

Given two nodes $\alpha = (l, r, c)$ and $\beta = (l', r', c')$ then β is **below** α, or α is **above** β, if $(l, r, c) < (l', r', c')$ in the lexicographic order.

The set of multipartitions of n becomes a poset ordered by **dominance** where $\boldsymbol{\lambda}$ dominates $\boldsymbol{\mu}$, or $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, if

$$\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda^{(l)}_j \ge \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu^{(l)}_j,$$

for $1 \le l \le \ell$ and $i \ge 1$. If $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ then we write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$. Let $\mathcal{P}^\Lambda_n = (\mathcal{P}^\Lambda_n, \trianglerighteq)$ be the poset of multipartitions of n ordered by dominance and let $(\mathcal{P}^\Lambda_n, \trianglelefteq)$ be the opposite poset.

Fix a multipartition $\boldsymbol{\lambda}$. Then a **$\boldsymbol{\lambda}$-tableau** is a bijective map $\mathfrak{t} \colon [\![\boldsymbol{\lambda}]\!] \longrightarrow \{1, 2, \dots, n\}$, which we identify with a labelling of $[\![\boldsymbol{\lambda}]\!]$ by $\{1, 2, \dots, n\}$. For example,

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 8 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|c|}\hline 9 & 10 & 11 \\ \hline 12 & 13 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\right) \quad\text{and}\quad \left(\begin{array}{|c|c|c|}\hline 9 & 12 & 13 \\ \hline 10 & \multicolumn{2}{c}{} \\ \cline{1-1} 11 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|}\hline 6 & 8 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \,\middle|\, \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\right)$$

are both $\boldsymbol{\lambda}$-tableaux when $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$ as above. In this way we speak of the rows, columns and components of tableaux. If $\mathfrak{t}$ is a tableau and $1 \le k \le n$ set $\operatorname{comp}_{\mathfrak{t}}(k) = l$ if k appears in the lth component of $\mathfrak{t}$.

A $\boldsymbol{\lambda}$-tableau is **standard** if its entries increase along rows and columns in each component. Both of the tableaux above are standard. Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of

standard $\boldsymbol{\lambda}$-tableaux and let $\operatorname{Std}(n) = \bigcup_{\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda}\operatorname{Std}(\boldsymbol{\lambda})$. Similarly set $\operatorname{Std}^2(\boldsymbol{\lambda}) = \{\,(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ and $\operatorname{Std}^2(\mathcal{P}_n^\Lambda) = \{\,(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda\,\}$.

If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau set $\operatorname{Shape}(\mathfrak{t}) = \boldsymbol{\lambda}$ and let $\mathfrak{t}_{\downarrow m}$ be the subtableau of $\mathfrak{t}$ which contains the numbers $\{1,2,\dots,m\}$. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau then $\operatorname{Shape}(\mathfrak{t}_{\downarrow m})$ is a multipartition for all $m\ge 0$. We extend the dominance ordering to the set of all standard tableaux by defining $\mathfrak{s}\trianglerighteq\mathfrak{t}$ if

$$\operatorname{Shape}(\mathfrak{s}_{\downarrow m}) \trianglerighteq \operatorname{Shape}(\mathfrak{t}_{\downarrow m}),$$

for $1\le m\le n$. As before, we write $\mathfrak{s}\triangleright\mathfrak{t}$ if $\mathfrak{s}\trianglerighteq\mathfrak{t}$ and $\mathfrak{s}\ne\mathfrak{t}$. We extend the dominance ordering to $\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ by declaring that $(\mathfrak{s},\mathfrak{t})\trianglerighteq(\mathfrak{u},\mathfrak{v})$ if $\mathfrak{s}\trianglerighteq\mathfrak{u}$ and $\mathfrak{t}\trianglerighteq\mathfrak{v}$. Similarly, $(\mathfrak{s},\mathfrak{t})\triangleright(\mathfrak{u},\mathfrak{v})$ if $(\mathfrak{s},\mathfrak{t})\trianglerighteq(\mathfrak{u},\mathfrak{v})$ and $(\mathfrak{s},\mathfrak{t})\ne(\mathfrak{u},\mathfrak{v})$

It is easy to see that there are unique standard $\boldsymbol{\lambda}$-tableaux $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}_{\boldsymbol{\lambda}}$ such that $\mathfrak{t}^{\boldsymbol{\lambda}}\trianglerighteq\mathfrak{t}\trianglerighteq\mathfrak{t}_{\boldsymbol{\lambda}}$, for all $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. The tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ has the numbers $1,2,\dots,n$ entered in order from left to right along the rows of $\mathfrak{t}^{\lambda^{(1)}}$, and then $\mathfrak{t}^{\lambda^{(2)}},\dots,\mathfrak{t}^{\lambda^{(\ell)}}$ and similarly, $\mathfrak{t}_{\boldsymbol{\lambda}}$ is the tableau with the numbers $1,\dots,n$ entered in order down the columns of $\mathfrak{t}^{\lambda^{(\ell)}},\dots,\mathfrak{t}^{\lambda^{(2)}},\mathfrak{t}^{\lambda^{(1)}}$. When $\boldsymbol{\lambda}=(3,1^2|2,1|3,2)$ then the two $\boldsymbol{\lambda}$-tableaux displayed above are $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}_{\boldsymbol{\lambda}}$.

Given a standard $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ define $d(\mathfrak{t})\in\mathfrak{S}_n$ to be the permutation such that $\mathfrak{t}=\mathfrak{t}^{\boldsymbol{\lambda}}d(\mathfrak{t})$. Let $\le$ be the Bruhat order on $\mathfrak{S}_n$ with the convention that $1\le w$ for all $w\in\mathfrak{S}_n$. By a well-known result of Ehresmann and James, if $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ then $\mathfrak{s}\trianglerighteq\mathfrak{t}$ if and only if $d(\mathfrak{s})\le d(\mathfrak{t})$; see, for example, [24, Theorem 3.8].

Recall from Section 2.1 that we have fixed a multicharge $\kappa\in\mathbb{Z}^\ell$. The **residue** of the node $A=(l,r,c)$ is $\operatorname{res}(A)=\kappa_l+c-r \pmod e$ (where we adopt the convention that $i\equiv i \pmod 0$, for $i\in\mathbb{Z}$). Thus, $\operatorname{res}(A)\in I$. A node A is an **i-node** if $\operatorname{res}(A)=i$. If $\mathfrak{t}$ is a $\boldsymbol{\mu}$-tableaux and $1\le k\le n$ then the **residue** of k in $\mathfrak{t}$ is $\operatorname{res}_{\mathfrak{t}}(k)=\operatorname{res}(A)$, where $A\in\boldsymbol{\mu}$ is the unique node such that $\mathfrak{t}(A)=k$. The **residue sequence** of $\mathfrak{t}$ is

$$\operatorname{res}(\mathfrak{t}) = (\operatorname{res}_{\mathfrak{t}}(1),\operatorname{res}_{\mathfrak{t}}(2),\dots,\operatorname{res}_{\mathfrak{t}}(n))\in I^n.$$

As an important special case we set $\mathbf{i}^\mu=\operatorname{res}(\mathfrak{t}^{\boldsymbol{\mu}})$, for $\boldsymbol{\mu}\in\mathcal{P}_n^\Lambda$.

Refine the dominance ordering on the set of standard tableaux by defining $\mathfrak{s}\,\underline{\blacktriangleright}\,\mathfrak{t}$ if $\mathfrak{s}\trianglerighteq\mathfrak{t}$ and $\operatorname{res}(\mathfrak{s})=\operatorname{res}(\mathfrak{t})$. Similarly, we write $(\mathfrak{s},\mathfrak{t})\,\underline{\blacktriangleright}\,(\mathfrak{u},\mathfrak{v})$ if $(\mathfrak{s},\mathfrak{t})\trianglerighteq(\mathfrak{u},\mathfrak{v})$, $\operatorname{res}(\mathfrak{s})=\operatorname{res}(\mathfrak{u})$ and $\operatorname{res}(\mathfrak{t})=\operatorname{res}(\mathfrak{v})$ and $(\mathfrak{s},\mathfrak{t})\blacktriangleright(\mathfrak{u},\mathfrak{v})$ now has the obvious meaning.

Following Brundan, Kleshchev and Wang [8, Definition. 3.5] we now define the degree of a standard tableau. Suppose that $\boldsymbol{\mu}\in\mathcal{P}_n^\Lambda$. A node A is an **addable node** of $\boldsymbol{\mu}$ if $A\notin\boldsymbol{\mu}$ and $\boldsymbol{\mu}\cup\{A\}$ is (the diagram of) a multipartition of $n+1$. Similarly, a node B is a **removable node** of $\boldsymbol{\mu}$ if $B\in\boldsymbol{\mu}$ and $\boldsymbol{\mu}\setminus\{B\}$ is a multipartition of $n-1$. Suppose that A is an i-node and define integers

$$d_A(\boldsymbol{\mu}) = \#\left\{\begin{matrix}\text{addable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly below } A\end{matrix}\right\} - \#\left\{\begin{matrix}\text{removable } i\text{-nodes of } \boldsymbol{\mu}\\ \text{strictly below } A\end{matrix}\right\}.$$

If $\mathfrak{t}$ is a standard $\boldsymbol{\mu}$-tableau define its **degree** inductively by setting $\deg_e(\mathfrak{t})=0$, if $n=0$, and if $n>0$ then

$$\deg_e(\mathfrak{t}) = \deg_e(\mathfrak{t}_{\downarrow(n-1)}) + d_A(\boldsymbol{\mu}), \tag{2.6}$$

where $A=\mathfrak{t}^{-1}(n)$. When e is understood we write $\deg(\mathfrak{t})$.

The following result shows that the degrees of the standard tableau are almost completely determined by the Cartan matrix (c_{ij}) of Γ_e.

2.7. **Lemma** (Brundan, Kleshchev and Wang [8, Proposition 3.13])**.** *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux such that $\mathfrak{s}\triangleright\mathfrak{t}=\mathfrak{s}(r,r+1)$, where $1\le r<n$ and $\mathbf{i}\in I^n$. Let $\mathbf{i}=\operatorname{res}(\mathfrak{s})$. Then $\deg_e(\mathfrak{s})=\deg_e(\mathfrak{t})+c_{i_ri_{r+1}}$.*

2.5. **The Murphy basis and cyclotomic Specht modules.** The cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ is a cellular algebra with several different cellular bases. This section introduces one of these bases, the Murphy basis, and uses it to define the Specht modules and simple modules of $\mathcal{H}_n^\Lambda$.

Fix a multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Following [10, Definition 3.14] and [4, §6], if $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ define $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} m_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}$, where $m_{\boldsymbol{\lambda}} = u_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}$ where

$$u_{\boldsymbol{\lambda}} = \prod_{1 \le l < \ell} \prod_{r=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(l)}|} \xi^{-\kappa_{l+1}}(L_r - [\kappa_{l+1}]) \quad \text{and} \quad x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w.$$

2.8. **Theorem** ([10, Theorem 3.26] and [4, Theorem 6.3])**.** *The cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ is free as a $\mathcal{O}$-module with cellular basis*

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$$

with respect to the weight poset $(\mathcal{P}_n^\Lambda, \trianglerighteq)$.

Proof. This theorem can be proved uniformly in all cases by modifying the argument of [10, Theorem 3.26], however, for future reference we explain how to deduce this result from the literature for the degenerate and non-degenerate algebras.

First suppose that $\xi = 1$. Then the element $m_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$, coincides exactly with the corresponding elements defined for the non-degenerate cyclotomic Hecke algebras in [4, §6]. It follows that $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathcal{P}_n^\Lambda \,\}$ is the Murphy basis of the degenerate cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ defined in [4, §6] and that the theorem is just a restatement of [4, Theorem 6.3] when $\xi = 1$.

Now suppose that $\xi \ne 1$ and, as in Remark 2.3, let $L_r' = (\xi - 1)L_r + 1$ be the 'non-degenerate' Jucys-Murphy elements for $\mathcal{H}_n^\Lambda$, for $1 \le r \le n$. An application of the definitions shows that if $\kappa \in \mathbb{Z}$ then

$$\xi^{-\kappa}(L_r - [\kappa]) = \frac{\xi^{-\kappa}}{\xi - 1}(L_r' - \xi^\kappa).$$

Therefore, $u_{\boldsymbol{\lambda}}$ is a scalar multiple of the element $u_{\boldsymbol{\lambda}}^+$ given by [10, Definition 3.1,3.5]. Consequently, if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $m_{\mathfrak{s}\mathfrak{t}}$ is a scalar multiple of the corresponding Murphy basis element from [10, Definition 3.14]. Hence, the theorem is an immediate consequence of [10, Theorem 3.26] in the non-degenerate case. □

Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. The (cyclotomic) **Specht module** $\underline{S}^{\boldsymbol{\lambda}}$ is the cell module associated to $\boldsymbol{\lambda}$ using the (ungraded) cellular basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$. We underline $\underline{S}^{\boldsymbol{\lambda}}$ to emphasise that $\underline{S}^{\boldsymbol{\lambda}}$ is not graded. When $\mathcal{O}$ is a field let $\underline{D}^{\boldsymbol{\lambda}} = \underline{S}^{\boldsymbol{\lambda}} / \operatorname{rad} \underline{S}^{\boldsymbol{\lambda}}$ and set $\mathcal{K}_n^\Lambda = \{\, \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \mid \underline{D}^{\boldsymbol{\lambda}} \ne 0 \,\}$. Ariki [2] has given a combinatorial description of the set $\mathcal{K}_n^\Lambda$. By the theory of cellular algebras [12], $\{\, \underline{D}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_n^\Lambda$-modules.

The following well-known fact is fundamental to all of the results in this paper.

2.9. **Lemma.** *Suppose that $1 \le r < n$ and that $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then*

$$m_{\mathfrak{s}\mathfrak{t}} L_r \equiv [c_r(\mathfrak{t})] m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \triangleright \mathfrak{t} \\ \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \pmod{\mathcal{H}_n^{\triangleright \boldsymbol{\lambda}}},$$

for some $r_{\mathfrak{v}} \in \mathcal{O}$.

Proof. If $\xi = 1$ then this is a restatement of [4, Lemma 6.6]. If $\xi \ne 1$ then

$$m_{\mathfrak{s}\mathfrak{t}} L_r' = \xi^{c_r(\mathfrak{t})} m_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \triangleright \mathfrak{t}} r_{\mathfrak{v}}' m_{\mathfrak{s}\mathfrak{t}} \qquad (\text{mod } \mathcal{H}_n^{\triangleright \boldsymbol{\lambda}}),$$

for some $r_v' \in \mathcal{O}$, by [17, Proposition 3.7] (and the notational translations given in the proof of Theorem 2.8). As $L_r = (L_r' - 1)/(\xi - 1)$ the result follows. □

2.6. **Cyclotomic quiver Hecke algebras.** Brundan and Kleshchev [6] have given a very different presentation of $\mathcal{H}_n^\Lambda$. This presentation is more difficult to work with but it has the advantage of showing that $\mathcal{H}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra.

2.10. **Definition** (Brundan-Kleshchev [6]). *Suppose that $n \ge 0$ and $e \in \{0, 2, 3, 4, \dots\}$. The **cyclotomic quiver Hecke algebra**, or **cyclotomic Khovanov-Lauda–Rouquier algebra**, of weight Λ and type Γ_e is the unital associative $\mathcal{O}$-algebra $\mathcal{R}_n^\Lambda = \mathcal{R}_n^\Lambda(\mathcal{O})$ with generators*

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$$

and relations

$$y_1^{(\Lambda,\alpha_{i_1})} e(\mathbf{i}) = 0, \qquad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) = 1,$$

$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$

$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r\psi_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \qquad y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \tag{2.11}$$

$$\begin{aligned}
\psi_r y_s &= y_s \psi_r, && \text{if } s \ne r, r+1,\\
\psi_r \psi_s &= \psi_s \psi_r, && \text{if } |r-s| > 1,
\end{aligned} \tag{2.12}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1},\\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \to i_{r+1},\\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1},\\ (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1}\\ e(\mathbf{i}), & \text{otherwise,} \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ \big(\psi_{r+1}\psi_r\psi_{r+1} + y_r - 2y_{r+1} + y_{r+2}\big)e(\mathbf{i}), & \\ & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{i}), & \text{otherwise,} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in I^n$ and all admissible r and s. Moreover, $\mathcal{R}_n^\Lambda$ is naturally $\mathbb{Z}$-graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg y_r = 2 \qquad \textit{and} \qquad \deg \psi_s e(\mathbf{i}) = -c_{i_s, i_{s+1}},$$

for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$.

2.13. *Remark.* The presentation of $\mathcal{R}_n^\Lambda$ given in Definition 2.10 differs by a choice of signs with the definition given in [6, Theorem 1.1]. The presentation of $\mathcal{R}_n^\Lambda$ given above agrees with that used in [22] as the orientation of the quiver is reversed in [22].

The connection between the cyclotomic quiver Hecke algebras of type Γ_e and the cyclotomic Hecke algebras of type $G(\ell, 1, n)$ is given by the following remarkable result of Brundan and Kleshchev.

2.14. **Theorem** (Brundan-Kleshchev's graded isomorphism theorem [6, Theorem 1.1]). *Suppose that $\mathcal{O} = K$ is a field, $\xi \in K$ as above, and that $\Lambda = \Lambda(\boldsymbol{\kappa})$. Then there is an isomorphism of algebras $\underline{\mathcal{R}}_n^\Lambda \cong \mathcal{H}_n^\Lambda$.*

We prove a stronger version of Theorem 2.14 in Theorem 4.32 below. For now we note the following simple corollary of Theorem 2.14. Recall that a choice of multicharge $\boldsymbol{\kappa}$ determines a dominant weight $\Lambda_e(\boldsymbol{\kappa})$.

2.15. **Corollary.** *Suppose that $n \ge 0$, $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ and that*

$$e > \max\{\, n + \kappa_k - \kappa_l \mid 1 \le k, l \le \ell \,\}.$$

Fix invertible scalars $\xi_0 \in K$ and $\xi_e \in K$ such that ξ_0 is not a root of unity and ξ_e is a primitive eth root of unity. Then the cyclotomic Hecke algebras $\mathcal{H}^{\Lambda_0(\kappa)}_{K,\xi_0}$ and $\mathcal{H}^{\Lambda_e(\boldsymbol{\kappa})}_{\mathscr{K},\xi_e}$ are isomorphic $\mathbb{Z}$-graded K-algebras.

Proof. Let $\mathcal{R}^\Lambda_n(0) \cong \mathcal{H}_n(K,\xi_0,\boldsymbol{\kappa})$ and $\mathcal{R}^\Lambda_n(e) \cong \mathcal{H}_n(K,\xi_e,\boldsymbol{\kappa})$ be the corresponding cyclotomic quiver Hecke algebras as in Theorem 2.14. By [15, Lemma 4.1], $e(\mathbf{i}) \neq 0$ if and only if $\mathbf{i} = \operatorname{res}(\mathfrak{t})$, for some standard tableau $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}^\Lambda_n)$. The definition of e ensures that if $\mathbf{i} = \mathbf{i}^\mathfrak{t}$ then $i_r = i_{r+1}$ or $i_r = i_{r+1} \pm 1$ if and only if $i_r \equiv i_{r+1} \pmod{e}$ or $i_r \equiv i_{r+1} \pm 1 \pmod{e}$. Therefore, $\mathcal{R}^\Lambda_n(0) \cong \mathcal{R}^\Lambda_n(e)$ arguing directly from the presentations of the cyclotomic quiver Hecke algebras given in Definition 2.10. Hence, the result follows by Theorem 2.14. □

Therefore, without loss of generality, we may assume that $e > 0$. In the appendix we show how to modify the results and definitions in this paper to cover the case when $e = 0$ directly.

Under the assumptions of the Corollary we note that the algebras $\mathcal{H}^\Lambda_{K,\xi_0}$ and $\mathcal{H}^\Lambda_{K,\xi_e}$ are Morita equivalent by the main result of [11]. That these algebras are actually isomorphic is another miracle provided by Brundan and Kleshchev's graded isomorphism theorem.

The following consequence of Theorem 2.14 will be needed later.

2.16. **Corollary.** *Suppose that $1 \le r \le n$ and $\mathbf{i} \in I^n$. Then $y_r^{N_r(\mathbf{i})} e(\mathbf{i}) = 0$, where $N_r(\mathbf{i}) = \#\operatorname{Std}(\mathbf{i})$.*

Proof. This is a well-known application of Theorem 2.14 and Lemma 2.9. □

3. Seminormal forms for Hecke algebras

In this chapter we develop the theory of seminormal forms in a slightly more general context than appears in the literature. In particular, in this paper a seminormal basis will be a basis for $\mathcal{H}^\Lambda_n$ rather than a basis of a Specht module of $\mathcal{H}^\Lambda_n$. We also treat all of the variations of the seminormal bases simultaneously as this will give us the flexibility to change seminormal forms when we use them in the next chapter to study the connections between $\mathcal{H}^\Lambda_n$ and the cyclotomic quiver Hecke algebra $\mathcal{R}^\Lambda_n$.

3.1. **Content functions and the Gelfand-Zetlin algebra.** Underpinning Brundan and Kleshchev's graded isomorphism theorem (Theorem 2.14) is the decomposition of any $\mathcal{H}^\Lambda_n$-module into a direct sum of generalised eigenspaces for the Jucys-Murphy elements $L_1, \dots, L_n$. This section studies the action of the Jucys-Murphy elements on $\mathcal{H}^\Lambda_n$. The results in this section are well-known, at least to experts, but they are needed in the sequel.

The **content** of the node $\gamma = (l, r, c)$ is the integer

$$\mathrm{c}_\gamma = \kappa_l - r + c$$

If $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ is a standard $\boldsymbol{\lambda}$-tableau and $1 \le k \le n$ then the **content** of k in $\mathfrak{t}$ is $\mathrm{c}_k(\mathfrak{t}) = \mathrm{c}_\gamma$, where $\mathfrak{t}(\gamma) = k$ for $\gamma \in [\![\boldsymbol{\lambda}]\!]$.

3.1. **Definition.** *Let $\mathcal{O}$ be a commutative integral domain and suppose that $t \in \mathcal{O}^\times$ is an invertible element of $\mathcal{O}$. The pair $(\mathcal{O}, t)$* ***separates*** *$\operatorname{Std}(\mathcal{P}^\Lambda_n)$ if*

$$[n]^!_t \prod_{1 \le l < m \le \ell} \prod_{-n < d < n} [\kappa_l - \kappa_m + d]_t \in \mathcal{O}^\times.$$

Fix a multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ and let $\mathcal{H}^\Lambda_n(\mathcal{O})$ be the Hecke algebra defined over $\mathcal{O}$ with parameter t. In spite of our notation, note that $\mathcal{H}^\Lambda_n(\mathcal{O})$ depends only on $\boldsymbol{\kappa}$ and

not directly on $\Lambda = \Lambda_e(\boldsymbol{\kappa})$. Let $\mathscr{K}$ be a field which contains the field of fractions of $\mathcal{O}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_\mathcal{O} \mathscr{K}$.

Throughout this chapter we are going to work with the Hecke algebras $\mathcal{H}_n^\Lambda(\mathcal{O})$ and $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_\mathcal{O} \mathscr{K}$, however, we have in mind the situation of Theorem 2.14. By assumption $e > 0$, so we can replace the multicharge $\boldsymbol{\kappa}$ with $(\kappa_1 + a_1 e, \kappa_2 + a_2 e, \dots, \kappa_\ell + a_\ell e)$, for any integers $a_1, \dots, a_\ell \in \mathbb{Z}$, without changing the dominant weight $\Lambda = \Lambda_e(\boldsymbol{\kappa})$. In view of Definition 3.1 we therefore assume that

$$\kappa_l - \kappa_{l+1} \geq n, \qquad \text{for } 1 \leq l < \ell. \tag{3.2}$$

Until further notice, we fix a multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ satisfying (3.2) and consider the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ with parameter t.

Although we do not need this, we remark that it follows from [1] and [4, Theorem 6.11] that $\mathcal{H}_n^\Lambda(\mathscr{K}, t)$ is semisimple if and only if $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$. Our main use of the separation condition is the following fundamental fact which is easily proved by induction on n; see, for example, [17, Lemma 3.12].

3.3. **Lemma.** *Suppose that $\mathcal{O}$ is an integral domain and $t \in \mathcal{O}^\times$ is invertible. Then $(\mathcal{O}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ if and only if*

$$\mathfrak{s} = \mathfrak{t} \quad \text{if and only if} \quad [c_r(\mathfrak{s})] = [c_r(\mathfrak{t})] \text{ for } 1 \leq r \leq n,$$

for all $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$.

Following [28], define the **Gelfand-Zetlin subalgebra** of $\mathcal{H}_n^\Lambda$ to be the algebra $\mathscr{L}(\mathcal{O}) = \langle L_1, \dots, L_n \rangle$. The aim of this section is to understand the semisimple representation theory of $\mathscr{L} = \mathscr{L}(\mathcal{O})$. It follows from Definition 2.2 that $\mathscr{L}$ is a commutative subalgebra of $\mathcal{H}_n^\Lambda$.

If $\mathcal{O}$ is an integral domain then it follows from Lemma 2.9 that, as an $(\mathscr{L}, \mathscr{L})$-bimodule, $\mathcal{H}_n^\Lambda(\mathcal{O})$ has a composition series with composition factors which are $\mathcal{O}$-free of rank 1 upon which L_r acts as multiplication by $c_r(\mathfrak{s})$ from the left and as multiplication by $c_r(\mathfrak{t})$ from the right. Obtaining a better description of $\mathscr{L}$, and of $\mathcal{H}_n^\Lambda$ as an $(\mathscr{L}, \mathscr{L})$-bimodule, in the non-semisimple case is likely to be important. For example, the dimension of $\mathscr{L}$ over a field is not known in general.

3.4. **Proposition** (cf. [3, Proposition 3.17])**.** *Suppose that $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$, where $\mathscr{K}$ is a field and $0 \neq t \in \mathscr{K}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a semisimple $\big(\mathscr{L}, \mathscr{L}\big)$-bimodule with decomposition*

$$\mathcal{H}_n^\Lambda(\mathscr{K}) = \bigoplus_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \\ \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})}} H_{\mathfrak{s}\mathfrak{t}},$$

where $H_{\mathfrak{s}\mathfrak{t}} = \{\, h \in \mathcal{H}_n^\Lambda \mid L_r h = [c_r(\mathfrak{s})]h \text{ and } hL_r = [c_r(\mathfrak{t})]h, \text{ for } 1 \leq r \leq n \,\}$ is one dimensional.

Proof. By Lemma 2.9, the Jucys-Murphy elements $L_1, \dots, L_n$ are a family of JM-elements for $\mathcal{H}_n^\Lambda$ in the sense of [26, Definition 2.4]. Therefore, the result is a special case of [26, Theorem 3.7]. □

Key to the proof of the results in [26] are the following elements which have their origins in the work of Murphy [27]. For $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ define

$$F_\mathfrak{t} = \prod_{k=1}^{n} \prod_{\substack{c \in \mathscr{E} \\ [c_k(\mathfrak{t})] \neq [c]}} \frac{L_k - [c]}{[c_k(\mathfrak{t})] - [c]} \tag{3.5}$$

where $\mathscr{E} = \{\, c_r(\mathfrak{t}) \mid 1 \leq r \leq n \text{ and } \mathfrak{t} \in \operatorname{Std}(n) \,\}$ is the set of the possible contents that can appear in a standard tableau of size n. By definition, $F_\mathfrak{t} \in \mathscr{L}(\mathscr{K})$ and it follows directly from Proposition 3.4 that if $h_{\mathfrak{u}\mathfrak{v}} \in H_{\mathfrak{u}\mathfrak{v}}$ then

$$F_\mathfrak{s} h_{\mathfrak{u}\mathfrak{v}} F_\mathfrak{t} = \delta_{\mathfrak{s}\mathfrak{u}} \delta_{\mathfrak{v}\mathfrak{t}} h_{\mathfrak{s}\mathfrak{t}}, \tag{3.6}$$

for all $(\mathfrak{s},\mathfrak{t}),(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Therefore, $H_{\mathfrak{s}\mathfrak{t}}=F_\mathfrak{s}\mathcal{H}_n^\Lambda F_\mathfrak{t}$.

By Proposition 3.4 we can write $1=\sum_{\mathfrak{s},\mathfrak{t}}e_{\mathfrak{s}\mathfrak{t}}$ for unique $e_{\mathfrak{s}\mathfrak{t}}\in H_{\mathfrak{s}\mathfrak{t}}$. Since $F_\mathfrak{t}=F_\mathfrak{t}^*$, the last displayed equation implies that $F_\mathfrak{t}=e_{\mathfrak{t}\mathfrak{t}}\in H_{\mathfrak{t}\mathfrak{t}}$ is an idempotent. Consequently,

$$\mathscr{L}(\mathscr{K})=\bigoplus_{\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)}H_{\mathfrak{t}\mathfrak{t}}=\bigoplus_{\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)}\mathscr{K}F_\mathfrak{t}.$$

In particular, $F_\mathfrak{t}$ is a primitive idempotent in $\mathscr{L}(\mathscr{K})$. If follows that $\mathscr{L}(\mathscr{K})$ is a split semisimple algebra of dimension $\#\operatorname{Std}(\mathcal{P}_n^\Lambda)$.

3.2. **Seminormal forms.** Seminormal bases for $\mathcal{H}_n^\Lambda$ are well-known in the literature, having their origins in the work of Young from [34]. Many examples of "seminormal bases" appear in the literature. In this section we classify the seminormal bases of $\mathcal{H}_n^\Lambda$. This characterisation of seminormal forms appears to be new, even in the special of the symmetric groups, although some of the details will be familiar to experts.

Throughout this section we assume that $\mathscr{K}$ is a field, $0\neq t\in\mathscr{K}$ and that $(\mathscr{K},t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$. Recall the decomposition $\mathcal{H}_n^\Lambda=\bigoplus_{(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)}H_{\mathfrak{s}\mathfrak{t}}$ from Proposition 3.4.

Define an **involution** on an algebra A to be an algebra anti-automorphism of A of order 2.

3.7. **Definition.** *Suppose that $(\mathscr{K},t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and let ι be an involution on $\mathcal{H}_n^\Lambda(\mathscr{K})$. An **$\iota$-seminormal basis** of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a basis of the form $\{\,f_{\mathfrak{s}\mathfrak{t}}\mid f_{\mathfrak{s}\mathfrak{t}}=\iota(f_{\mathfrak{t}\mathfrak{s}})\in H_{\mathfrak{s}\mathfrak{t}}$ for $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)\,\}$.*

For now we take $*$ to be the unique anti-isomorphism of $\mathcal{H}_n^\Lambda(\mathscr{K})$ which fixes each of the generators $T_1,\dots,T_{n-1},L_1,\dots,L_n$ of Definition 2.2. Then $m_{\mathfrak{s}\mathfrak{t}}^*=m_{\mathfrak{t}\mathfrak{s}}$, for all $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. The assumption that $f_{\mathfrak{s}\mathfrak{t}}^*=f_{\mathfrak{t}\mathfrak{s}}$ is not essential for what follows but it is natural because we want to work within the framework of cellular algebras.

In order to describe the action of $\mathcal{H}_n^\Lambda$ on its seminormal bases, if $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$ then define the integers

$$\rho_r(\mathfrak{t})=c_r(\mathfrak{t})-c_{r+1}(\mathfrak{t}),\qquad\text{for }1\le r<n.\tag{3.8}$$

Then $\rho_r(\mathfrak{t})$ is the 'axial distance' between r and $r+1$ in the tableau $\mathfrak{t}$.

A **$*$-seminormal coefficient system** for $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a set of scalars

$$\boldsymbol{\alpha}=\{\,\alpha_r(\mathfrak{s})\mid 1\le r<n\text{ and }\mathfrak{s}\in\operatorname{Std}(n)\,\}$$

in $\mathscr{K}$ such that if $1\le r<n$ and $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$ then

$$\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}s_r)\alpha_r(\mathfrak{t}s_rs_{r+1})=\alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}s_{r+1})\alpha_{r+1}(\mathfrak{t}s_{r+1}s_r)\tag{3.9}$$

and, setting $\mathfrak{v}=\mathfrak{t}(r,r+1)$, then $\alpha_r(\mathfrak{t})=0$ if $\mathfrak{v}\notin\operatorname{Std}(\boldsymbol{\lambda})$ and otherwise

$$\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{v})=\frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}.\tag{3.10}$$

We will see that condition (3.9) ensures that the braid relations of length 3 are satisfied by $T_1,\dots,T_{n-1}$ and that (3.10) corresponds to the quadratic relations. Quite surprisingly, as the proof of Theorem 3.22 below shows, (3.10) also encodes the KLR grading on $\mathcal{H}_n^\Lambda$.

Usually, we omit the $*$ and simply call $\boldsymbol{\alpha}$ a seminormal coefficient system.

3.11. **Example** A nice 'rational' seminormal coefficient system is given by

$$\alpha_r(\mathfrak{t})=\begin{cases}\frac{[1+\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{t})]}, & \text{if }\mathfrak{t}(r,r+1)\text{ is standard,}\\ 0, & \text{otherwise,}\end{cases}$$

for $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$. ◇

3.12. **Example** By Proposition 3.18 below, the following seminormal coefficient system is associated with the Murphy basis of $\mathcal{H}_n^\Lambda$: if $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ set $\mathfrak{v} = \mathfrak{t}(r, r+1)$ and define

$$\alpha_r(\mathfrak{t}) = \begin{cases} 1 & \text{if } \mathfrak{v} \text{ is standard and } \mathfrak{t} \rhd \mathfrak{v}, \\ \frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}, & \text{if } \mathfrak{v} \text{ is standard and } \mathfrak{v} \rhd \mathfrak{t}, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le r < n$. ◇

Another seminormal coefficient system, which is particularly well adapted to Brundan and Kleshchev's Graded Isomorphism Theorem 2.14, is given in Section 5.1.

3.13. **Lemma.** *Suppose that $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a seminormal basis of $\mathcal{H}_n^\Lambda$. Then there exists a unique seminormal coefficient system $\boldsymbol{\alpha}$ such that if $1 \le r \le n$ and $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then*

$$f_{\mathfrak{s}\mathfrak{t}} T_r = \alpha_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}},$$

where $\mathfrak{v} = \mathfrak{t}(r, r+1)$.

Proof. The uniqueness statement is automatic, since $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$, so we need to prove that such a seminormal coefficient system $\boldsymbol{\alpha}$ exists.

Fix $(\mathfrak{s}, t) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$ and write

$$f_{\mathfrak{s}\mathfrak{t}} T_r = \sum_{(\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)} a_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}},$$

for some $a_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$. Multiplying on the left by $F_{\mathfrak{s}}$ it follows that $a_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $\mathfrak{u} = \mathfrak{s}$. If $k \neq r, r+1$ then L_k commutes with T_r so it follows $a_{\mathfrak{s}\mathfrak{v}} \neq 0$ only if $[c_k(\mathfrak{v})] = [c_k(\mathfrak{t})]$, for $k \neq r, r+1$. Using Definition 3.1, and arguing as in Lemma 3.3, this implies that $a_{\mathfrak{s}\mathfrak{v}} \neq 0$ only if $\mathfrak{v} \in \{\mathfrak{t}, \mathfrak{t}(r, r+1)\}$. Therefore, we can write

$$f_{\mathfrak{s}\mathfrak{t}} T_r = \alpha_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} + \alpha'_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{t}},$$

for some $\alpha_r(\mathfrak{t}), \alpha'_r(\mathfrak{t}) \in \mathscr{K}$, where $\mathfrak{v} = \mathfrak{t}(r, r+1)$. (Here, and below, we adopt the convention that $f_{\mathfrak{s}\mathfrak{v}} = 0$ if either of $\mathfrak{s}$ or $\mathfrak{v}$ is not standard.) By Definition 2.2, $T_r L_r = L_{r+1}(T_r - t + 1) - 1$, so multiplying both sides of the last displayed equation on the right by L_r and comparing the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ on both sides shows that

$$[c_{r+1}(\mathfrak{t})]\big(\alpha'_r(\mathfrak{t}) - t + 1\big) - 1 = \alpha'_r(\mathfrak{t})[c_r(\mathfrak{t})].$$

Hence, $\alpha'_r(\mathfrak{t}) = -1/[\rho_r(\mathfrak{t})]$ as claimed. If $\mathfrak{v}$ is not standard then we set $\alpha_r(\mathfrak{t}) = 0$. If $\mathfrak{v}$ is standard then comparing the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ on both sides of

$$\Big(\alpha_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}}\Big) T_r = f_{\mathfrak{s}\mathfrak{t}} T_r^2 = f_{\mathfrak{s}\mathfrak{t}}\big((t-1)T_r + t\big)$$

shows that $\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{v}) = \frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}$ in accordance with (3.10).

Finally, it remains to show that (3.9) holds. If $1 \le r < n$ then $T_r T_{r+1} T_r = T_{r+1} T_r T_{r+1}$ by Definition 2.2. On the other hand, if we set $\mathfrak{t}_1 = \mathfrak{t}(r, r+1)$, $\mathfrak{t}_2 = \mathfrak{t}(r+1, r+2)$, $\mathfrak{t}_{12} = \mathfrak{t}_1(r+1, r+2)$ $\mathfrak{t}_{21} = \mathfrak{t}_2(r, r+1)$ and $\mathfrak{t}_{121} = \mathfrak{t}_{212} = \mathfrak{t}(r, r+2)$

then direct calculation shows that $0 = f_{\mathfrak{s}\mathfrak{t}}(T_rT_{r+1}T_r - T_{r+1}T_rT_{r+1})$ is equal to

$$\begin{aligned}
-\Big(&\frac{1}{[\rho_r(\mathfrak{t})]^2[\rho_{r+1}(\mathfrak{t})]} - \frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]^2} + \frac{\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{t}_1)}{[\rho_{r+1}(\mathfrak{t}_1)]} - \frac{\alpha_{r+1}(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_2)}{[\rho_r(\mathfrak{t}_2)]}\Big)f_{\mathfrak{s}\mathfrak{t}}\\
&+\alpha_r(\mathfrak{t})\Big(\frac{1}{[\rho_r(\mathfrak{t}_1)][\rho_{r+1}(\mathfrak{t}_1)]} + \frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]} - \frac{1}{[\rho_{r+1}(\mathfrak{t})][\rho_{r+1}(\mathfrak{t}_1)]}\Big)f_{\mathfrak{s}\mathfrak{t}_1}\\
&-\alpha_{r+1}(\mathfrak{t})\Big(\frac{1}{[\rho_r(\mathfrak{t}_2)][\rho_{r+1}(\mathfrak{t}_2)]} + \frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]} - \frac{1}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{t}_2)]}\Big)f_{\mathfrak{s}\mathfrak{t}_2}\\
&-\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_1)\Big(\frac{1}{[\rho_r(\mathfrak{t}_{12})]} - \frac{1}{[\rho_{r+1}(\mathfrak{t})]}\Big)f_{\mathfrak{s}\mathfrak{t}_{12}}\\
&+\alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}_2)\Big(\frac{1}{[\rho_{r+1}(\mathfrak{t}_{21})]} - \frac{1}{[\rho_r(\mathfrak{t})]}\Big)f_{\mathfrak{s}\mathfrak{t}_{21}}\\
&+\big(\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_1)\alpha_r(\mathfrak{t}_{12}) - \alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}_2)\alpha_{r+1}(\mathfrak{t}_{21})\big)f_{\mathfrak{s}\mathfrak{t}_{121}}.
\end{aligned}$$

By our conventions, if any tableau $\mathfrak{t}_?$ is not standard then $f_{\mathfrak{s}\mathfrak{t}_?}$ and the corresponding $\boldsymbol{\alpha}$-coefficient are both zero. As the coefficient of $f_{\mathfrak{s}\mathfrak{t}_{121}}$ in the last displayed equation is zero it follows that (3.9) holds. Consequently, $\boldsymbol{\alpha} = \{\alpha_r(\mathfrak{t})\}$ is a seminormal coefficient system, completing the proof. (It is not hard to see, using (3.10) and identities like $\rho_r(\mathfrak{t}_1) = -\rho_r(\mathfrak{t})$ and $\rho_r(\mathfrak{t}_{12}) = \rho_{r+1}(\mathfrak{t})$, that the remaining coefficients in the last displayed equation are automatically zero.) ∎

Lemma 3.13 really says that acting from the right on a seminormal basis determines a seminormal coefficient system. Similarly, the left action on a seminormal basis determines a seminormal coefficient system. In general, the seminormal coefficient systems attached to the left and right actions will be different, however, because we are assuming that our seminormal bases are $*$-invariant these left and right coefficient systems coincide. Thus, for $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$ we also have $T_rf_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} - \frac{1}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{u} = \mathfrak{s}(r, r+1)$.

Exactly as eigenvectors are not uniquely determined by their eigenvalues, seminormal bases are not uniquely determined by seminormal coefficient systems. We now fully characterize seminormal bases — and prove a converse to Lemma 3.13.

Recall that a set of idempotents in an algebra is **complete** if they sum to 1.

3.14. **Theorem** (The Seminormal Basis Theorem). *Suppose that $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and that $\boldsymbol{\alpha}$ is a seminormal coefficient system for $\mathcal{H}_n^\Lambda(\mathscr{K})$. Then $\mathcal{H}_n^\Lambda(\mathscr{K})$ has a $*$-seminormal basis $\{ f_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\lambda}) \}$ such that if $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then*

$$f_{\mathfrak{s}\mathfrak{t}}^* = f_{\mathfrak{t}\mathfrak{s}}, \quad f_{\mathfrak{s}\mathfrak{t}}L_k = [c_k(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{t}} \quad \text{and} \quad f_{\mathfrak{s}\mathfrak{t}}T_r = \alpha_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}}, \tag{3.15}$$

where $\mathfrak{v} = \mathfrak{t}(r, r+1)$ and $f_{\mathfrak{s}\mathfrak{v}} = 0$ if $\mathfrak{v}$ is not standard. Moreover, there exist non-zero scalars $\gamma_{\mathfrak{t}} \in \mathscr{K}$, for $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$, such that

$$F_{\mathfrak{u}}f_{\mathfrak{s}\mathfrak{t}}F_{\mathfrak{v}} = \delta_{\mathfrak{u}\mathfrak{s}}\delta_{\mathfrak{t}\mathfrak{v}}f_{\mathfrak{s}\mathfrak{t}}, \quad f_{\mathfrak{s}\mathfrak{t}}f_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}\mathfrak{u}}\gamma_{\mathfrak{t}}f_{\mathfrak{s}\mathfrak{v}}, \quad \text{and} \quad F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}}f_{\mathfrak{t}\mathfrak{t}}. \tag{3.16}$$

Furthermore, $\{ F_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \}$ is a complete set of pairwise orthogonal primitive idempotents. In particular, every irreducible $\mathcal{H}_n^\Lambda(\mathscr{K})$-module is isomorphic to $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$, for some $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$, and $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K}) \cong F_{\mathfrak{u}}\mathcal{H}_n^\Lambda(\mathscr{K})$ if and only if $\operatorname{Shape}(\mathfrak{s}) = \operatorname{Shape}(\mathfrak{u})$.

Finally, the basis $\{ f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$ is uniquely determined by the choice of seminormal coefficient system $\boldsymbol{\alpha}$ and the scalars $\{ \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \} \subseteq \mathscr{K}^\times$.

Proof. For each $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ fix an arbitrary pair of tableaux and a non-zero element $f_{\mathfrak{s}\mathfrak{t}} \in H_{\mathfrak{s}\mathfrak{t}}$. Then $f_{\mathfrak{s}\mathfrak{t}}$ is a simultaneous eigenvector for all of the elements of $\mathscr{L}$, where they act from the left and from the right.

Now, suppose that $1 \le r < n$ and that $\mathfrak{v} = \mathfrak{t}(r, r+1)$ is standard. Then $\alpha_r(\mathfrak{t}) \neq 0$ so we can set $f_{\mathfrak{s}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{s}\mathfrak{t}}(T_r + \frac{1}{[\rho_r(\mathfrak{t})]})$. Equivalently, $f_{\mathfrak{s}\mathfrak{t}}T_r = \alpha_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}}$. Then using the relations in $\mathcal{H}_n^\Lambda(\mathscr{K})$ and the defining properties of the seminormal coefficient system $\boldsymbol{\alpha}$, it is straightforward to check that $f_{\mathfrak{s}\mathfrak{v}} L_k = [c_k(\mathfrak{v})] f_{\mathfrak{s}\mathfrak{v}}$, so that $f_{\mathfrak{s}\mathfrak{v}} \in H_{\mathfrak{s}\mathfrak{v}}$. Moreover, $f_{\mathfrak{s}\mathfrak{v}} \neq 0$ since $f_{\mathfrak{s}\mathfrak{t}} = \frac{1}{\alpha_r(\mathfrak{v})} f_{\mathfrak{s}\mathfrak{v}}(T_r + \frac{1}{[\rho_r(\mathfrak{v})]})$.

More generally, it is easy to see that if $\mathfrak{v}$ is any $\boldsymbol{\lambda}$-tableau then there is a sequence of standard tableaux $\mathfrak{v}_0 = \mathfrak{s}, \mathfrak{v}_1, \dots, \mathfrak{v}_z = \mathfrak{v}$ such that $\mathfrak{v}_{i+1} = \mathfrak{v}_i(r_i, r_i + 1)$, for some integers $1 \le r_i < n$. Therefore, continuing in this way it follows that given two tableaux $\mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ we can define non-zero elements $f_{\mathfrak{u}\mathfrak{v}} \in H_{\mathfrak{u}\mathfrak{v}}$ which satisfy (3.15). It follows that, once $f_{\mathfrak{s}\mathfrak{t}}$ is fixed, there is at most one choice of elements $\{\, f_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$, such that (3.15) holds.

To complete the proof that the seminormal coefficient system determines a seminormal basis we need to check that the elements $f_{\mathfrak{u}\mathfrak{v}}$ from the last paragraph are well-defined. That is, we need to show that $f_{\mathfrak{u}\mathfrak{v}}$ is independent of the choice of the sequences of simple transpositions which link $\mathfrak{u}$ and $\mathfrak{v}$ to $\mathfrak{s}$ and $\mathfrak{t}$, respectively. Equivalently, we need to prove that the action of $\mathcal{H}_n^\Lambda(\mathscr{K})$ given by (3.15) respects the relations of $\mathcal{H}_n^\Lambda(\mathscr{K})$. Using (3.15), all of the relations in Definition 2.2 are easy to check except for the braid relations of length three which hold by virtue of the argument of Lemma 3.13. Hence, by choosing elements $f_{\mathfrak{s}\mathfrak{t}} \in H_{\mathfrak{s}\mathfrak{t}}$, for $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$, the seminormal coefficient system determines a unique seminormal basis.

Using (3.6) it is straightforward to prove (3.16) so we leave these details to the reader; cf. [26, Theorem 3.16]. In particular, this shows that $F_{\mathfrak{s}} = \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}$ is an idempotent. To show that $F_{\mathfrak{s}}$ is primitive, suppose that a is a non-zero element of $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$. By (3.15), $a = \sum_{\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})} r_{\mathfrak{v}} f_{\mathfrak{s}\mathfrak{v}}$, for some $r_{\mathfrak{v}} \in \mathscr{K}$. Fix $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ such that $r_{\mathfrak{t}} \neq 0$. Then $f_{\mathfrak{s}\mathfrak{t}} = 1/r_{\mathfrak{t}} a F_{\mathfrak{t}} \in F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$. Using (3.15) we deduce that $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$ has basis $\{\, f_{\mathfrak{s}\mathfrak{v}} \mid \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$. Consequently, $a\mathcal{H}_n^\Lambda = F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$, showing that $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$ is irreducible. Therefore, $F_{\mathfrak{s}}$ is a primitive idempotent in $\mathcal{H}_n^\Lambda(\mathscr{K})$.

The last paragraph, together with (3.10), implies that if $\mathfrak{s}, \mathfrak{u} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda \cong F_{\mathfrak{u}}\mathcal{H}_n^\Lambda$ where an isomorphism is given by $f_{\mathfrak{s}\mathfrak{t}} \mapsto f_{\mathfrak{u}\mathfrak{t}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Consequently, if $\mathfrak{s}$ and $\mathfrak{u}$ are standard tableaux of different shape then $F_{\mathfrak{s}}\mathcal{H}_n^\Lambda \ncong F_{\mathfrak{u}}\mathcal{H}_n^\Lambda$ because the multiplicity of $S^{\boldsymbol{\lambda}} \cong F_{\mathfrak{s}}\mathcal{H}_n^\Lambda(\mathscr{K})$ in $\mathcal{H}_n^\Lambda(\mathscr{K})$ is $\#\operatorname{Std}(\boldsymbol{\lambda})$ by the Wedderburn theorem.

Finally, it remains to show that the basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is uniquely determined by $\boldsymbol{\alpha}$ and the choice of the γ-coefficients $\{\, \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$. If $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then we have shown that, once $f_{\mathfrak{s}\mathfrak{t}}$ is fixed, there is a unique seminormal basis $\{\, f_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$ satisfying (3.15). In particular, taking $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$ and fixing $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ determines these basis elements. By (3.16) the choice of $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ also uniquely determines $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$. Conversely, by setting $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} F_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ for any choice of non-zero scalars $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \in \mathscr{K}$, for $\boldsymbol{\lambda} \in \mathscr{K}$, the seminormal coefficient system $\boldsymbol{\alpha}$ determines a unique seminormal basis. ☐

The results which follow are independent of the choice of seminormal coefficient system $\boldsymbol{\alpha}$, however, the choice of γ-coefficients will be important — and in what follows it will be useful to be able to vary both the seminormal coefficient system $\boldsymbol{\alpha}$ and the γ-coefficients.

The proof of Theorem 3.14 implies that the choice of $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ determines all of the scalars $\gamma_{\mathfrak{s}}$, for $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. In what follows we need the following result which makes the relationship between these coefficients more explicit.

3.17. **Corollary.** *Suppose that $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and that $\mathfrak{v} = \mathfrak{t}(r, r+1)$ is standard, where $1 \le r < n$. Then $\alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}} = \alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}}$.*

Proof. Applying (3.15) and (3.16) several times each,

$$\begin{aligned}\gamma_{\mathfrak{v}} f_{\mathfrak{v}\mathfrak{v}} &= f_{\mathfrak{v}\mathfrak{v}} f_{\mathfrak{v}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}}\Big(T_r + \frac{1}{[\rho_r(\mathfrak{t})]}\Big) f_{\mathfrak{v}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}} T_r f_{\mathfrak{v}\mathfrak{v}}\\ &= \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}}\Big(\alpha_r(\mathfrak{v}) f_{\mathfrak{t}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{v})]} f_{\mathfrak{v}\mathfrak{v}}\Big) = \frac{\alpha_r(\mathfrak{v})}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}} f_{\mathfrak{t}\mathfrak{v}}\\ &= \frac{\alpha_r(\mathfrak{v})}{\alpha_r(\mathfrak{t})} \gamma_{\mathfrak{t}} f_{\mathfrak{v}\mathfrak{v}}.\end{aligned}$$

Comparing coefficients, $\alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}} = \alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}}$ as required. □

3.3. **Seminormal bases and the Murphy basis.** In this section we compute the Gram determinant of the Specht modules of $\mathcal{H}_n^\Lambda$, with respect to the Murphy basis, as a product of primes. These determinants are already explicitly known [4, 16–18] but all existing formulas describe them as products of rational functions, or of rational numbers in the degenerate case.

By Theorem 2.8, the Murphy basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis for $\mathcal{H}_n^\Lambda$ over an arbitrary ring. In this section we continue to work with the generic Hecke algebra $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{O})$ with parameter t and multicharge $\boldsymbol{\kappa}$ satisfying (3.2).

As $(\mathscr{K}, \mathfrak{t})$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$, for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ we can define

$$f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}.$$

By Lemma 2.9, $f_{\mathfrak{s}\mathfrak{t}} \equiv m_{\mathfrak{s}\mathfrak{t}} + \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$ where $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$. It follows that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ in the sense of Definition 3.7.

3.18. **Proposition.** *The basis $\{\, f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$ is the $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ determined by the seminormal coefficient system defined in Example 3.12 and the choices*

$$\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = [\boldsymbol{\lambda}]_t^! \prod_{1\le l<m\le \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m],$$

for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

Proof. This is equivalent to [25, Theorem 2.11] in the non-degenerate case and to [4, Proposition 6.8] in the degenerate case, however, rather than translating the notation from these two papers it is easier to prove this directly.

As noted above, $(\mathcal{O}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $f_{\mathfrak{s}\mathfrak{t}} \equiv m_{\mathfrak{s}\mathfrak{t}} + \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$ where $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$. Therefore, in view of (3.16), $\{\, f_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \,\}$ is a $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. By Theorem 3.14, this basis is determined by a seminormal coefficient system $\boldsymbol{\alpha}$ and by a choice of scalars $\{\, \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$. If $\mathfrak{t} \rhd \mathfrak{v} = \mathfrak{t}(r, r+1)$ then, by definition, $m_{\mathfrak{s}\mathfrak{t}} T_r = m_{\mathfrak{s}\mathfrak{v}}$. The transition matrix between the $\{m_{\mathfrak{s}\mathfrak{t}}\}$ and $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is unitriangular so, in view of Theorem 3.14, $f_{\mathfrak{s}\mathfrak{t}} T_r = f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}}$. Therefore, by (3.10), the seminormal coefficient system corresponding to the basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is the one appearing in Example 3.12.

It remains to determine the scalars $\{\, \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$ corresponding to $\{f_{\mathfrak{s}\mathfrak{t}}\}$. It is well-known, and easy to prove using the relations in $\mathcal{H}_n^\Lambda$, that $x_{\boldsymbol{\lambda}}^2 = [\boldsymbol{\lambda}]_t^! x_{\boldsymbol{\lambda}}$. Therefore, by Lemma 2.9,

$$f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}^2 \equiv [\boldsymbol{\lambda}]_t^! m_{\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}} \equiv [\boldsymbol{\lambda}]_t^! \prod_{1\le l<m\le \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m] \cdot m_{\boldsymbol{\lambda}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}.$$

Hence, $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = [\boldsymbol{\lambda}]_t^! \prod_{1\le l<m\le \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m]$ by (3.16). □

As noted after Theorem 2.8, the Murphy basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \,\}$ of $\mathcal{H}_n^\Lambda$ gives a basis $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$ of each Specht module $\underline{S}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. For

example, we can set $m_{\mathfrak{t}} = m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. By (2.5), the cellular basis equips each Specht module $S^{\boldsymbol{\lambda}}$ with an inner product $\langle\ ,\ \rangle$. The matrix

$$\underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \big(\langle m_{\mathfrak{s}}, m_{\mathfrak{t}}\rangle\big)_{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})}$$

is the **Gram matrix** of $S^{\boldsymbol{\lambda}}$ with respect to the Murphy basis. Similarly, the seminormal basis yields a second basis $\{\, f_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$ of $S^{\boldsymbol{\lambda}}(\mathscr{K})$, where $f_{\mathfrak{t}} = m_{\mathfrak{t}}F_{\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The transition matrix between these two bases is unitriangular, so by (3.16) we have

$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \det\big(\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle\big) = \prod_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \gamma_{\mathfrak{t}}. \tag{3.19}$$

This 'classical' formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ is well-known as it is the cornerstone used to prove the classical formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ as a rational function in [17, Theorem 3.35]. The following definition will allow us to give an 'integral' closed formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$.

3.20. **Definition.** *Suppose that $e \in \{0, 2, 3, 4, \dots\}$, p is a prime integer and that $\boldsymbol{\lambda} \in \mathcal{P}_n^{\Lambda}$ is a multipartition of n. Define*

$$\deg_e(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \deg_e \mathfrak{t} \qquad \textit{and} \qquad \operatorname{Deg}_p(\boldsymbol{\lambda}) = \sum_{k\geq 1} \deg_{p^k}(\boldsymbol{\lambda}).$$

By definition, $\deg_e(\boldsymbol{\lambda})$ and $\operatorname{Deg}_p(\boldsymbol{\lambda})$ are integers which, *a priori*, could be positive, negative or zero. In fact, the next result shows that they are always non-negative integers, although we do not known of a direct combinatorial proof of this. By definition, the integers $\deg_e(\boldsymbol{\lambda})$ and $\operatorname{Deg}_p(\boldsymbol{\lambda})$ depend on $\boldsymbol{\kappa}$ and e. Our definitions ensure that the tableau degrees $\deg_e(\mathfrak{t})$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, coincide with (2.6) when $\Lambda = \Lambda_e(\boldsymbol{\kappa})$.

For $k \in \mathbb{N}$, let $\Phi_k = \Phi_k(t)$ be the kth cyclotomic polynomial in t. As is well-known, these polynomials are pairwise distinct irreducible polynomials in $\mathbb{Z}[t]$ and

$$[n] = \prod_{1<d|n} \Phi_d(t), \tag{3.21}$$

whenever $n \geq 1$.

3.22. **Theorem.** *Suppose that $\kappa_l - \kappa_{l+1} > n$, for $1 \leq l < \ell$, and that $\mathcal{O} = \mathbb{Z}[t, t^{-1}]$. Then*

$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = t^{\ell(\boldsymbol{\lambda})} \prod_{e\geq 2} \Phi_e(t)^{\deg_e(\boldsymbol{\lambda})},$$

where $\ell(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \ell(d(\mathfrak{t}))$.

Proof. As remarked above, $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \prod_{\mathfrak{t}} \gamma_{\mathfrak{t}}$. Therefore, to prove the theorem it is enough to show that if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$\gamma_{\mathfrak{t}} = t^{\ell(d(\mathfrak{t}))} \prod_{e>1} \Phi_e(t)^{\deg_e(\mathfrak{t})}.$$

We prove this by induction on the dominance ordering.

Suppose first that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$. Then Proposition 3.18 gives an explicit formula for $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ and, using (2.6), it is straightforward to check by induction on n that our claim is true in this case. Suppose then that $\mathfrak{t}^{\boldsymbol{\lambda}} \rhd \mathfrak{t}$. Then we can write $\mathfrak{t} = \mathfrak{s}(r, r+1)$ for some $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ such that $\mathfrak{s} \rhd \mathfrak{t}$, and where $1 \leq r < n$. Therefore, using induction, Corollary 3.17 and the seminormal coefficient system of Proposition 3.18,

$$\gamma_{\mathfrak{t}} = t^{\ell(d(\mathfrak{s}))} \frac{[1+\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{t})]} \prod_{e>1} \Phi_e(t)^{\deg_e(\mathfrak{s})}.$$

By definition, $[k] = -t^k[-k]$, for any $k \in \mathbb{Z}$. Now $\rho_r(\mathfrak{s}) = -\rho_r(\mathfrak{t}) > 0$ by (3.2), so
$$\frac{[1+\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{t})]} = t\,\frac{[1+\rho_r(\mathfrak{s})][-\rho_r(\mathfrak{t})-1]}{[\rho_r(\mathfrak{s})][-\rho_r(\mathfrak{t})]} = t\prod_{e>1}\Phi_e(t)^{d_e},$$
where, according to (3.21), the integer d_e is given in terms of the quiver Γ_e by
$$d_e = \begin{cases} -2, & \text{if } i_r = i_{r+1},\\ 2, & \text{if } i_r \leftrightarrows i_{r+1},\\ 1, & \text{if } i_r \leftarrow i_{r+1} \text{ or } i_r \rightarrow i_{r+1},\\ 0, & \text{otherwise.}\end{cases}$$
Applying Lemma 2.7 now completes the proof of our claim — and hence proves the theorem. □

3.23. *Remark.* We can remove the factor $t^{\ell(\boldsymbol{\lambda})}$ from Theorem 3.22 by rescaling the generators $T_1, \dots, T_{n-1}$ so that the quadratic relations in Definition 2.2 become $(T_r - t^{\frac12})(T_r + t^{-\frac12})$, for $1 \le r < n$. Note that the integer d_e in the proof of Theorem 3.14 is equal to the degree of the homogeneous generator $\psi_r e(\mathbf{i})$ in the cyclotomic KLR algebra $\mathcal{R}_n^\Lambda$.

Setting $t = 1$ gives the degenerate cyclotomic Hecke algebras. As a special case, the next result gives an integral closed formula for the Gram determinants of the Specht modules of the symmetric groups.

3.24. **Corollary.** *Suppose that $\kappa_l - \kappa_{l+1} > n$, for $1 \le l < \ell$, and that $\mathcal{O} = \mathbb{Z}$ and $t = 1$. Then*
$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \prod_{1<p \text{ prime}} p^{\operatorname{Deg}_p(\boldsymbol{\lambda})},$$
for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

Proof. This follows by setting $t = 1$ in Theorem 3.22 and using the following well-known property of the cyclotomic polynomials:
$$\Phi_e(1) = \begin{cases} p, & \text{if } e = p^k \text{ for some } k \ge 1,\\ 1, & \text{otherwise.}\end{cases}$$
□

3.25. **Corollary.** *Suppose that $e \in \{0, 2, 3, 4, 5, \dots\}$ and that $p > 0$ is an integer prime. Then $\deg_e(\boldsymbol{\lambda}) \ge 0$ and $\operatorname{Deg}_p(\boldsymbol{\lambda}) \ge 0$, for all $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.*

Proof. As the Murphy basis is defined over $\mathbb{Z}[t, t^{-1}]$, the Gram determinant $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ belongs to $\mathbb{Z}[t, t^{-1}]$. Therefore, $\deg_e(\boldsymbol{\lambda}) \ge 0$ whenever $e > 1$ by Theorem 3.22. Consequently, $\operatorname{Deg}_p(\boldsymbol{\lambda}) \ge 0$. Finally, if $e \gg 0$ then $\deg_0(\mathfrak{t}) = \deg_e(\mathfrak{t})$ for any $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$, so $\deg_e(\boldsymbol{\lambda}) \ge 0$ for $e \in \{0, 2, 3, 4, \dots\}$ as claimed. □

The statement of Corollary 3.25 is purely combinatorial so it should have a direct combinatorial proof. We sketch a second representation theoretic proof of this result which suggests that a combinatorial proof may be difficult.

A **graded set** is a set D equipped with a **degree** function $\deg : D \longrightarrow \mathbb{Z}$. Define
$$\deg D = \sum_{d \in D} \deg d \in \mathbb{Z}.$$
If D is a graded set and $z \in \mathbb{Z}$ let $q^z D$ be the graded set where an element $d \in D$ has degree $z + \deg d$. More generally, if $f(q) \in \mathbb{N}[q, q^{-1}]$ let $f(q)D$ be the graded set which is the disjoint union of the appropriate number of shifted copies of D. For example $(2 + q)D = D \sqcup D \sqcup qD$.

If $e \in \{0, 2, 3, 4, \dots\}$ let $\mathrm{Std}_e(\boldsymbol{\lambda})$ be the graded set $\mathrm{Std}(\boldsymbol{\lambda})$ equipped with the degree function $\mathfrak{t} \mapsto \deg_e(\mathfrak{t})$, for $\mathfrak{t} \in \mathrm{Std}_e(\boldsymbol{\lambda})$.

Fix $e \in \{0, 2, 3, 4, \dots\}$ and consider the Hecke algebra $\mathcal{H}_n^\Lambda(\mathbb{C})$ over $\mathbb{C}$ with parameter ξ, a primitive eth root of unity if $e > 0$ or a non-root of unity if $e = 0$. Let $S^{\boldsymbol{\lambda}}$ be the graded Specht module introduced in [8] (see Section 5.2), and let $D^{\boldsymbol{\mu}} = S^{\boldsymbol{\mu}} / \operatorname{rad} S^{\boldsymbol{\mu}}$ be the graded simple quotient of $S^{\boldsymbol{\mu}}$, as in [15]. Let $\mathcal{K}_n^\Lambda$ be the set of **Kleshchev multipartitions** so that $\{\, D^{\boldsymbol{\mu}}\langle k\rangle \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \text{ and } k \in \mathbb{Z} \,\}$ is a complete set of non-isomorphic graded simple $\mathcal{H}_n^\Lambda$-modules. As recalled in Section 5.2, $S^{\boldsymbol{\lambda}}$ comes equipped with a homogeneous basis $\{\, \psi_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$. Let $d_{\boldsymbol{\lambda}\mu}(q) = [S^{\boldsymbol{\lambda}}{:}D^{\boldsymbol{\mu}}]_q$ be the corresponding graded decomposition number.

Fix a total ordering $\prec$ on $\mathrm{Std}(\boldsymbol{\lambda})$ which extends the dominance ordering. By Gaussian elimination, there exists a graded subset $\mathrm{DStd}_e(\boldsymbol{\lambda})$ of $\mathrm{Std}(\boldsymbol{\lambda})$ and a basis $\{\, C_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{DStd}_e(\boldsymbol{\lambda}) \,\}$ such that $C_{\mathfrak{t}} = \psi_{\mathfrak{t}} + \sum_{\mathfrak{v} \prec \mathfrak{t}} c_{\mathfrak{t}\mathfrak{v}} \psi_{\mathfrak{v}} + \operatorname{rad} S^{\boldsymbol{\lambda}}$, for some $c_{\mathfrak{t}\mathfrak{v}} \in \mathbb{C}$ such that $c_{\mathfrak{t}\mathfrak{v}} \neq 0$ only if $\deg \mathfrak{v} = \deg \mathfrak{t}$ and $\operatorname{res}(\mathfrak{v}) = \operatorname{res}(\mathfrak{t})$. In particular, $\operatorname{Dim} D^{\boldsymbol{\lambda}} = \deg \mathrm{DStd}_e(\boldsymbol{\lambda})$. Repeating this argument with each factor of the radical filtration of $S^{\boldsymbol{\lambda}}$, it follows that there exists a bijection of graded sets

$$\Theta_{\boldsymbol{\lambda}} : \mathrm{Std}_e(\boldsymbol{\lambda}) \xrightarrow{\ \sim\ } \bigsqcup_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\, \mathrm{DStd}_e(\boldsymbol{\mu}).$$

Now if $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$ then $D^{\boldsymbol{\mu}} \cong (D^{\boldsymbol{\mu}})^{\circledast}$, so that $\deg \mathrm{DStd}_e(\boldsymbol{\mu}) = 0$. It follows that $\deg q^z\, \mathrm{DStd}_e(\boldsymbol{\mu}) = z \dim \underline{D}^{\boldsymbol{\mu}}$, for $z \in \mathbb{Z}$. Therefore, using the bijection $\Theta_{\boldsymbol{\lambda}}$,

$$\deg_e(\boldsymbol{\lambda}) = \deg \mathrm{Std}_e(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} \deg \Big(d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\, \mathrm{DStd}_e(\boldsymbol{\mu}) \Big) = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d'_{\boldsymbol{\lambda}\mu}(1) \dim \underline{D}^{\boldsymbol{\mu}},$$

where $d'_{\boldsymbol{\lambda}\mu}(1)$ is the derivative of the graded decomposition number $d_{\lambda\mu}(q)$ evaluated at $q = 1$. By [7, Corollary 5.15], $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q]$ when K is a field of characteristic zero, so we get that $\deg_e(\boldsymbol{\lambda}) \geq 0$ and hence this gives an alternative proof of Corollary 3.25.

In characteristic zero the graded cyclotomic Schur algebras are expected to be Koszul (this is true when $e = 0$ by [14, Theorem C]). This conjecture implies that the Jantzen and grading filtrations of the graded Weyl modules, and hence of the graded Specht modules, coincide. Therefore, Corollary 3.25 is compatible with this Koszulity Conjecture via Ryom-Hansen's [30, Theorem 1] description of the Jantzen sum formula; see also [35, Theorem 2.11].

The construction of the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$ given above is not unique because it involves many choices. It natural to ask if there is are natural choices for the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$ and the bijections $\Theta_{\boldsymbol{\lambda}}$ so that they correspond to a basis of $S^{\boldsymbol{\lambda}}$ which is uniquely determined in some way. For level 2 such bijections are implicit in [9, §9] when $e = 0$ and in [14, Appendix] for $e \geq n$. It is interesting to note that the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$, together with the bijections $\Theta_{\boldsymbol{\lambda}}$, determine the graded decomposition numbers because if $\mathfrak{s} \in \mathrm{DStd}_e(\boldsymbol{\mu})$ then

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \sum_{\mathfrak{t} \in \Theta_{\boldsymbol{\lambda}}^{-1}(\mathfrak{s})} q^{\deg \mathfrak{t} - \deg \mathfrak{s}},$$

where we abuse notation and let $\Theta_{\boldsymbol{\lambda}}^{-1}(\mathfrak{s})$ be the set of tableaux in $\mathrm{Std}(\boldsymbol{\lambda})$ which are mapped onto a (shifted) copy of $\mathfrak{s}$ by $\Theta_{\boldsymbol{\lambda}}$. In particular, we can take $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\mu}}$ because it is easy to see that $\mathfrak{t}^{\boldsymbol{\mu}} \in \mathrm{DStd}_e(\boldsymbol{\mu})$ whenever $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$. That is, our arguments prove the existence of a purely combinatorial formula for the parabolic Kazhdan-Lusztig polynomials $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$.

4. Integral Quiver Hecke algebras

The Seminormal Basis Theorem 3.14 compactly describes much of the semisimple representation theory of $\mathcal{H}_n^\Lambda(\mathscr{K})$. For symmetric groups, Murphy [27] showed that seminormal bases can also be used to study the non-semisimple representation theory. Murphy's ideas were extended to the cyclotomic Hecke algebras in [25, 26]. In this section we further extend Murphy's ideas to connect seminormal bases and the KLR grading on $\mathcal{H}_n^\Lambda$.

4.1. **Lifting idempotents.** As Section 3.2, we continue to assume that $\boldsymbol{\kappa}$ satisfies (3.2) and that $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$, where $\mathscr{K}$ is a field and $0 \neq t \in \mathscr{K}$. If $\mathcal{O}$ is a subring of $\mathscr{K}$ then we identify $\mathcal{H}_n^\Lambda(\mathcal{O})$ with the obvious $\mathcal{O}$-subalgebra of $\mathcal{H}_n^\Lambda(\mathscr{K})$ so that $\mathcal{H}_n^\Lambda(\mathscr{K}) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$ as $\mathscr{K}$-algebras.

Let $J(\mathcal{O})$ be the **Jacobson radical** of $\mathcal{O}$, the intersection of all of the maximal ideals of $\mathcal{O}$.

4.1. **Definition.** *Suppose that $\mathcal{O}$ is a subring of $\mathscr{K}$ and $t \in \mathcal{O}^\times$. Then $(\mathcal{O}, t)$, is an **e-idempotent subring** of $\mathscr{K}$ if the following hold:*

a) *$(\mathcal{O}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$;*
b) *$[k]_t$ is invertible in $\mathcal{O}$ whenever $k \not\equiv 0 \pmod{e}$, for $k \in \mathbb{Z}$; and*
c) *$[k]_t \in J(\mathcal{O})$ whenever $k \in e\mathbb{Z}$.*

When e and t are understood, we simply call $\mathcal{O}$ an idempotent subring. Note that if $\mathscr{K}$ contains the field of fractions of $\mathcal{O}$ then Definition 4.1(a) ensures that $\mathcal{H}_n^\Lambda(\mathscr{K})$ is semisimple and has seminormal bases. We fix such a $*$-seminormal basis $\{f_{\mathfrak{st}}\}$, together with the corresponding seminormal coefficient system $\boldsymbol{\alpha}$ and γ-coefficients, until further notice.

Let $(\mathcal{O}, t)$ be an e-idempotent subring and suppose $c \not\equiv d \pmod{e}$, for $c, d \in \mathbb{Z}$. Then $[c] - [d] = t^d[c-d]$ is invertible in $\mathcal{O}$. We use this fact below without mention.

4.2. **Examples** The following local rings are all examples of idempotent subrings.

a) Suppose that $\mathscr{K} = \mathbb{Q}$ and $t = 1$. Then $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathcal{O} = \mathbb{Z}_{(p)}$ is a p-idempotent subring of $\mathbb{Q}$ for any prime p.
b) Let K be any field and set $\mathscr{K} = K(x)$, where x is an indeterminate over K, and $t = x+\xi$, where ξ is a primitive eth root of unity in K. Then $\mathcal{O} = K[x]_{(x)}$ is an e-idempotent subring of $\mathscr{K}$.
c) Let $\mathscr{K} = \mathbb{Q}(x, \xi)$, where x is an indeterminate over $\mathbb{Q}$ and $\xi = \exp(2\pi i/e)$ is a primitive eth root of unity in $\mathbb{C}$. Let $t = x + \xi$. Then $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathcal{O} = \mathbb{Z}[x, \xi]_{(x)}$ is an e-idempotent subring of $\mathscr{K}$.
d) Maintain the notation of the last example and let $p > 1$ be a prime not dividing e. Let $\Phi_{e,p}(x)$ be a polynomial in $\mathbb{Z}[x]$ whose reduction modulo p is the minimum polynomial of a primitive eth root of unity in an algebraically closed field of characteristic p. Then $\mathcal{O} = \mathbb{Z}[x, \xi]_{(x,p,\Phi_{e,p}(\xi))}$ is an e-idempotent subring of $\mathbb{C}(x)$.

$\diamond$

Suppose that $\mathbf{i} \in I^n$ and set $\operatorname{Std}(\mathbf{i}) = \{\, \mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \mid \operatorname{res}(\mathfrak{t}) = \mathbf{i} \,\}$. Define the **residue idempotent** $f_{\mathbf{i}}^{\mathcal{O}}$ by

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} F_{\mathfrak{t}}. \tag{4.3}$$

By Theorem 3.14, $f_{\mathbf{i}}^{\mathcal{O}}$ is an idempotent in $\mathcal{H}_n^\Lambda(\mathscr{K})$. In the rest of this section, we fix a seminormal basis $\{f_{\mathfrak{st}}\}$ of $\mathcal{H}_n^\Lambda(\mathscr{K})$ which is determined by a seminormal coefficient system $\{\alpha_r(\mathfrak{s})\}$ and a choice of $\gamma_{\mathfrak{t}^\lambda}$. Then we have that $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}}$.

4.4. **Lemma.** *Suppose that $\mathcal{O}$ is an idempotent subring of $\mathscr{K}$ and that $\mathbf{i} \in I^n$. Then $f_{\mathbf{i}}^{\mathcal{O}} \in \mathscr{L}(\mathcal{O})$. In particular, $f_{\mathbf{i}}^{\mathcal{O}}$ is an idempotent in $\mathcal{H}_n^\Lambda(\mathcal{O})$.*

Proof. This result is proved when $\mathcal{O}$ is a discrete valuation ring in [26, Lemma 4.2], however, our weaker assumptions necessitate a different proof. Motivated, in part, by the proof of [27, Theorem 2.1], if $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$ define

$$F'_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{c\in\mathscr{C} \\ c_k(\mathfrak{t})\not\equiv c \pmod e}} \frac{L_k - [c]}{[c_k(\mathfrak{t})] - [c]}.$$

Since $\mathcal{O}$ is an e-idempotent subring, $F'_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}) \subset \mathcal{H}_n^\Lambda(\mathcal{O})$. By Theorem 3.14, $\sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)} F_{\mathfrak{s}}$ is the identity element of $\mathcal{H}_n^\Lambda(\mathscr{K})$ so, using (3.15), we see that

$$F'_{\mathfrak{t}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)} F'_{\mathfrak{t}} F_{\mathfrak{s}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)} a_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{s}},$$

where $a_{\mathfrak{s}\mathfrak{t}} = \prod_{k,c}([c_k(\mathfrak{s})]-[c])/([c_k(\mathfrak{t})]-[c]) \in \mathcal{O}$. In particular, $a_{\mathfrak{t}\mathfrak{t}} = 1$. If $\mathfrak{s} \notin \operatorname{Std}(\mathbf{i})$ then there exists an integer k such that $\operatorname{res}_k(\mathfrak{s}) \neq \operatorname{res}_k(\mathfrak{t})$, so $[c_k(\mathfrak{s})] - [c_k(\mathfrak{t})] \in \mathcal{O}^\times$ and $a_{\mathfrak{s}\mathfrak{t}} = 0$. Therefore, $F'_{\mathfrak{t}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{s}}$. Consequently, $f_{\mathbf{i}}^{\mathcal{O}} F'_{\mathfrak{t}} = F'_{\mathfrak{t}} = F'_{\mathfrak{t}} f_{\mathbf{i}}^{\mathcal{O}}$ by (3.16). Notice that $F'_{\mathfrak{t}} F'_{\mathfrak{s}} = F'_{\mathfrak{s}} F'_{\mathfrak{t}}$ because $\mathscr{L}(\mathscr{K})$ is a commutative subalgebra of $\mathcal{H}_n^\Lambda(\mathscr{K})$. Therefore,

$$\prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (f_{\mathbf{i}}^{\mathcal{O}} - F'_{\mathfrak{t}}) = f_{\mathbf{i}}^{\mathcal{O}} \qquad + \sum_{\substack{\mathfrak{t}_1,\dots,\mathfrak{t}_k\in\operatorname{Std}(\mathbf{i}) \\ \text{distinct with } k>0}} (-1)^k F'_{\mathfrak{t}_1} F'_{\mathfrak{t}_2} \dots F'_{\mathfrak{t}_k}.$$

On the other hand, since $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} F_{\mathfrak{s}}$ and $a_{\mathfrak{t}\mathfrak{t}} = 1$,

$$\prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (f_{\mathbf{i}}^{\mathcal{O}} - F'_{\mathfrak{t}}) = \prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i}) \\ \mathfrak{s}\neq\mathfrak{t}}} (1 - a_{\mathfrak{s}\mathfrak{t}}) F_{\mathfrak{s}} = 0,$$

because $F_{\mathfrak{s}} F_{\mathfrak{t}} = 0$ whenever $\mathfrak{s} \neq \mathfrak{t}$ by (3.16). Combining the last two equations,

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\substack{\mathfrak{t}_1,\dots,\mathfrak{t}_k\in\operatorname{Std}(\mathbf{i}) \\ \text{distinct with } k>0}} (-1)^{k+1} F'_{\mathfrak{t}_1} F'_{\mathfrak{t}_2} \dots F'_{\mathfrak{t}_k}.$$

In particular, $f_{\mathbf{i}}^{\mathcal{O}} \in \mathscr{L}(\mathcal{O})$ as we wanted to show. ∎

4.5. **Corollary.** *Suppose that $\mathcal{O}$ is an idempotent subring of $\mathscr{K}$. Then $\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n \,\}$ is a complete set of pairwise orthogonal idempotents in $\mathcal{H}_n^\Lambda(\mathcal{O})$.*

Proof. By Theorem 3.14, $\{\, F_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \,\}$ is a complete set of pairwise orthogonal idempotents in $\mathcal{H}_n^\Lambda(\mathscr{K})$. Hence, the result follows from Lemma 4.4. ∎

If $\phi \in \mathcal{O}[X_1, \dots, X_n]$ is a polynomial in indeterminates $X_1, \dots, X_n$ over $\mathcal{O}$ then set $\phi(L) = \phi(L_1, \dots, L_n) \in \mathscr{L}(\mathcal{O})$. If $\mathfrak{s}$ is a tableau let $\phi(\mathfrak{s}) = \phi([c_1(\mathfrak{s})], \dots, [c_n(\mathfrak{s})])$ be the scalar in $\mathcal{O}$ obtained by evaluating the polynomial ϕ on the contents of $\mathfrak{s}$; that is, setting $X_1 = [c_1(\mathfrak{s})], \dots, X_n = [c_n(\mathfrak{s})]$. Then, $\phi(L) f_{\mathfrak{s}\mathfrak{t}} = \phi(\mathfrak{s}) f_{\mathfrak{s}\mathfrak{t}}$, for all $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$.

Ultimately, the next result will allow us to 'renormalise' intertwiners of the residue idempotents $f_{\mathbf{i}}^{\mathcal{O}}$, for $\mathbf{i} \in I^n$, so that they depend only on e rather than on ξ.

4.6. **Proposition.** *Suppose that $\mathbf{i} \in I^n$ and $\phi \in \mathcal{O}[X_1, \dots, X_n]$ is a polynomial such that $\phi(\mathfrak{t})$ is invertible in $\mathcal{O}$, for all $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. Then*

$$f_{\mathbf{i}}^{\phi} = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{t})} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}).$$

In particular, $f_{\mathbf{i}}^{\phi} \in \mathcal{H}_n^\Lambda(\mathcal{O})$.

Proof. By assumption, $\phi(\mathfrak{s})$ is invertible in $\mathcal{O}$ for all $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$. In particular, $f_{\mathbf{i}}^{\phi}$ is a well-defined element of $\mathscr{L}(\mathscr{K})$. It remains to show that $f_{\mathbf{i}}^{\phi} \in \mathscr{L}(\mathcal{O})$.

As in Lemma 4.4, for each $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$ define

$$F'_{\mathfrak{t}} = \prod_{\substack{c\in\mathscr{E}\\ c_k(\mathfrak{t})\not\equiv c \pmod e}} \frac{L_k - [c]}{[c_k(\mathfrak{t})] - [c]} \in \mathscr{L}(\mathcal{O}),$$

and write $F'_{\mathfrak{t}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{s}}$ for some $a_{\mathfrak{s}\mathfrak{t}} \in \mathcal{O}$. Recall from the proof of Lemma 4.4 that $a_{\mathfrak{t}\mathfrak{t}} = 1$.

Motivated by the definition of $F'_{\mathfrak{t}}$, set $F^{\phi}_{\mathfrak{t}} = \frac{\phi(L)}{\phi(\mathfrak{t})} F'_{\mathfrak{t}}$. Then $F^{\phi}_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O})$ and

$$F^{\phi}_{\mathfrak{t}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} \frac{\phi(L)}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = F_{\mathfrak{t}} + \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{s}\neq\mathfrak{t}}} \frac{a_{\mathfrak{s}\mathfrak{t}}\phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}}$$

by (3.15). Consequently, $F^{\phi}_{\mathfrak{t}} f^{\mathcal{O}}_{\mathbf{i}} = F^{\phi}_{\mathfrak{t}} = f^{\mathcal{O}}_{\mathbf{i}} F^{\phi}_{\mathfrak{t}}$. The idempotents $\{\, F_{\mathfrak{s}} \mid \mathfrak{s} \in \operatorname{Std}(\mathbf{i}) \,\}$ are pairwise orthogonal, so

$$f^{\phi}_{\mathbf{i}} F^{\phi}_{\mathfrak{t}} = \Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{s})} F_{\mathfrak{s}} \Big)\Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{a_{\mathfrak{s}\mathfrak{t}}\phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}} \Big) = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{a_{\mathfrak{s}\mathfrak{t}}}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = \frac{1}{\phi(\mathfrak{t})} F'_{\mathfrak{t}}.$$

Therefore, $f^{\phi}_{\mathbf{i}} F^{\phi}_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O})$, for all $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. By (3.15), $f^{\phi}_{\mathbf{i}} f^{\mathcal{O}}_{\mathbf{i}} = f^{\phi}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} f^{\phi}_{\mathbf{i}}$, so this implies that $f^{\phi}_{\mathbf{i}}(f^{\mathcal{O}}_{\mathbf{i}} - F^{\phi}_{\mathfrak{t}}) \equiv f^{\phi}_{\mathbf{i}} \pmod{\mathscr{L}(\mathcal{O})}$. Hence, working modulo $\mathscr{L}(\mathcal{O})$,

$$f^{\phi}_{\mathbf{i}} \equiv f^{\phi}_{\mathbf{i}} \prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (f^{\mathcal{O}}_{\mathbf{i}} - F^{\phi}_{\mathfrak{t}}) = f^{\phi}_{\mathbf{i}} \prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{s}\neq\mathfrak{t}}} \frac{a_{\mathfrak{s}\mathfrak{t}}\phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = 0,$$

where the last equality follows using the orthogonality of the idempotents $F_{\mathfrak{s}}$ once again. Therefore, $f^{\phi}_{\mathbf{i}} \in \mathscr{L}(\mathcal{O})$, completing the proof. ∎

Let ϕ be a polynomial in $\mathcal{O}[X_1, \dots, X_n]$ satisfying the assumptions of Proposition 4.6. Then $\phi(L) f^{\phi}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} = f^{\phi}_{\mathbf{i}} \phi(L)$ by (3.15). Abusing notation, in this situation we write

$$\frac{1}{\phi(L)} f^{\mathcal{O}}_{\mathbf{i}} = f^{\phi}_{\mathbf{i}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{s})} F_{\mathfrak{s}} = f^{\mathcal{O}}_{\mathbf{i}} \frac{1}{\phi(L)} \in \mathscr{L}(\mathcal{O}).$$

Note that, either by direction calculation or because $\mathscr{L}$ is commutative, we are justified in writing $f^{\mathcal{O}}_{\mathbf{i}} \frac{1}{\phi(L)} = \frac{1}{\phi(L)} f^{\mathcal{O}}_{\mathbf{i}}$.

We need the following three special cases of Proposition 4.6. For $1 \le r < n$ define $M_r = 1 - L_r + tL_{r+1}$ and $M'_r = 1 + tL_r - L_{r+1}$, for $1 \le r < n$. Applying the definitions, if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}(\mathcal{P}^{\Lambda}_n)$ then

$$M_r f_{\mathfrak{s}\mathfrak{t}} = t^{c_r(\mathfrak{s})}[1 - \rho_r(\mathfrak{s})] f_{\mathfrak{s}\mathfrak{t}} \quad\text{and}\quad M'_r f_{\mathfrak{s}\mathfrak{t}} = t^{c_{r+1}(\mathfrak{s})}[1 + \rho_r(\mathfrak{s})] f_{\mathfrak{s}\mathfrak{t}}. \tag{4.7}$$

Our main use of Proposition 4.6 is the following application which corresponds to taking $\phi(L)$ be to $L_r - L_{r+1}$, M_r and M'_r, respectively.

4.8. **Corollary.** *Suppose that $\mathcal{O}$ is an e-idempotent subring, $1 \le r < n$ and $\mathbf{i} \in I^n$.*

a) *If $i_r \neq i_{r+1}$ then* $\displaystyle \frac{1}{L_r - L_{r+1}} f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}).$

b) *If $i_r \neq i_{r+1} + 1$ then* $\displaystyle \frac{1}{M_r} f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1 - \rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}).$

c) *If $i_r \neq i_{r+1} - 1$ then* $\displaystyle \frac{1}{M'_r} f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_{r+1}(\mathfrak{t})}}{[1 + \rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}).$

4.2. **Intertwiners.** By Theorem 2.14, if K is a field then the KLR generators of $\mathcal{H}_n^\Lambda(K)$ satisfy $\psi_r e(\mathbf{i}) = e(s_r\cdot\mathbf{i})\psi_r$. This section defines analogous elements in $\mathcal{H}_n^\Lambda(\mathcal{O})$ which intertwine the residue idempotents $f_{\mathbf{i}}^{\mathcal{O}}$, for $\mathbf{i}\in I^n$.

4.9. **Lemma.** *Suppose that $i_r = i_{r+1}$, for some $\mathbf{i}\in I^n$ and $1\le r<n$. Then $T_r f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} T_r$.*

Proof. This follows directly from the Seminormal Basis Theorem 3.14. In more detail, note that if $\mathfrak{t}\in\operatorname{Std}(\mathbf{i})$ then r and $r+1$ cannot appear in the same row or in the same column of $\mathfrak{t}$. Therefore,

$$T_r f_{\mathbf{i}}^{\mathcal{O}} - f_{\mathbf{i}}^{\mathcal{O}} T_r = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\gamma_{\mathfrak{t}}}\Big(T_r f_{\mathfrak{t}\mathfrak{t}} - f_{\mathfrak{t}\mathfrak{t}}T_r\Big) = \sum_{\substack{\mathfrak{t},\mathfrak{v}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{v}=\mathfrak{t}(r,r+1)}}\Big(\frac{\alpha_r(\mathfrak{t})}{\gamma_{\mathfrak{t}}} - \frac{\alpha_r(\mathfrak{v})}{\gamma_{\mathfrak{v}}}\Big) f_{\mathfrak{v}\mathfrak{t}},$$

by (3.15). By Corollary 3.17, if $\mathfrak{v}=\mathfrak{t}(r,r+1)$ then $\alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}} = \alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}}$. Hence, $T_r f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} T_r$ as claimed. ∎

4.10. *Remark.* In the special case of the symmetric groups, Ryom-Hansen [31, §3] has proved an analogue of Lemma 4.9.

Using (3.15), it is easy to verify that $T_r f_{\mathbf{i}}^{\mathcal{O}} \ne f_{\mathbf{j}}^{\mathcal{O}} T_r$ if $\mathbf{j} = s_r\cdot\mathbf{i}\ne\mathbf{i}$, for $1\le r<n$ and $\mathbf{i}\in I^n$. The following elements will allow us to correct for this.

4.11. **Lemma.** *Suppose that $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $1\le r<n$. Let $\mathfrak{u}=\mathfrak{s}(r,r+1)$. Then $(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]f_{\mathfrak{u}\mathfrak{t}}$.*

Proof. Using (3.15) we obtain

$$(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})\big([c_r(\mathfrak{s})] - [c_{r+1}(\mathfrak{s})]\big)f_{\mathfrak{u}\mathfrak{t}} = \alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]f_{\mathfrak{u}\mathfrak{t}},$$

where, as usual, we set $f_{\mathfrak{u}\mathfrak{t}}=0$ if $\mathfrak{u}$ is not standard. ∎

Applying the $*$-involution, $f_{\mathfrak{s}\mathfrak{t}}(T_rL_r - L_rT_r) = -\alpha_r(\mathfrak{t})t^{c_{r+1}(\mathfrak{t})}[\rho_r(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{v}}$, where $\mathfrak{v}=\mathfrak{t}(r,r+1)$.

4.12. **Lemma.** *Suppose that $i_r\ne i_{r+1}$, for some $\mathbf{i}\in I^n$ and $1\le r<n$. Set $\mathbf{j}=s_r\cdot\mathbf{i}$. Then $(T_rL_r - L_rT_r)f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r)$.*

Proof. By definition, $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\gamma_{\mathfrak{s}}}f_{\mathfrak{s}\mathfrak{s}}$ so, by Lemma 4.11,

$$\begin{aligned}(T_rL_r - L_rT_r)f_{\mathbf{i}}^{\mathcal{O}} &= \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\gamma_{\mathfrak{s}}}(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{s}}\\ &= \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{u}=\mathfrak{s}(r,r+1)\in\operatorname{Std}(\mathcal{P}_n^\Lambda)}}\frac{\alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}}f_{\mathfrak{u}\mathfrak{s}}.\end{aligned}$$

Note that if $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ and $\mathfrak{u}=\mathfrak{s}(r,r+1)$ is standard then $\mathfrak{s}\in\operatorname{Std}(\mathbf{j})$. Similarly,

$$f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r) = \sum_{\substack{\mathfrak{u}\in\operatorname{Std}(\mathbf{j})\\ \mathfrak{s}=\mathfrak{u}(r,r+1)\in\operatorname{Std}(\mathbf{i})}} -\frac{\alpha_r(\mathfrak{u})t^{c_{r+1}(\mathfrak{u})}[\rho_r(\mathfrak{u})]}{\gamma_{\mathfrak{u}}}f_{\mathfrak{u}\mathfrak{s}}.$$

By (3.15), the tableaux in $\operatorname{Std}(\mathbf{i})$ and $\operatorname{Std}(\mathbf{j})$ which have r and $r+1$ in the same row or in the same column do not contribute to the right hand sides of either of the last two equations. Moreover, the map $\mathfrak{s}\mapsto\mathfrak{u}=\mathfrak{s}(r,r+1)$ defines a bijection from the set of tableaux in $\operatorname{Std}(\mathbf{i})$ such that r and $r+1$ appear in different rows and columns to the set of tableaux in $\operatorname{Std}(\mathbf{j})$ which have r and $r+1$ in different rows and columns. In particular, $(T_rL_r - L_rT_r)f_{\mathbf{i}}^{\mathcal{O}}=0$ if and only if $f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r)=0$.

To complete the proof suppose that $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ and that $\mathfrak{u}=\mathfrak{s}(r,r+1)\in\operatorname{Std}(\mathbf{j})$. Now, $\alpha_r(\mathfrak{u})\gamma_{\mathfrak{s}}=\alpha_r(\mathfrak{s})\gamma_{\mathfrak{u}}$, by Corollary 3.17, and $\rho_r(\mathfrak{u})=-\rho_r(\mathfrak{s})$, by definition. So

$$\frac{-\alpha_r(\mathfrak{u})t^{c_{r+1}(\mathfrak{u})}[\rho_r(\mathfrak{u})]}{\gamma_{\mathfrak{u}}}=\frac{-\alpha_r(\mathfrak{s})t^{c_r(\mathfrak{s})}[-\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}}=\frac{\alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}}.$$

Hence, comparing the equations above, $(T_rL_r-L_rT_r)f^{\mathcal{O}}_{\mathbf{i}}=f^{\mathcal{O}}_{\mathbf{j}}(T_rL_r-L_rT_r)$ as required. □

Recall the definitions of M_r and M'_r from (4.7), for $1\le r<n$. We finish this section by giving the commutation relations for the elements M_r, M'_r, $(1+T_r)$ and $(T_rL_r-L_rT_r)$. These will be important later.

4.13. **Lemma.** *Suppose that* $1\le r<n$. *Then*

$$(T_rL_r-L_rT_r)M_r=M'_r(T_rL_r-L_rT_r)\quad\textit{and}\quad(T_r-t)M_r=M'_r(1+T_r).$$

Proof. Both formulas can be proved by applying the relations in Definition 2.2. Alternatively, suppose that $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ and set $\mathfrak{v}=\mathfrak{t}(r,r+1)$. Then, by (4.7) and Lemma 4.11,

$$\begin{aligned}f_{\mathfrak{s}\mathfrak{t}}(T_rL_r-L_rT_r)M_r&=-\alpha_r(\mathfrak{t})t^{2c_r(\mathfrak{v})}[\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{v}}\\&=f_{\mathfrak{s}\mathfrak{t}}M'_r(T_rL_r-L_rT_r),\end{aligned}$$

where the last equality follows because $c_r(\mathfrak{v})=c_{r+1}(\mathfrak{t})$ and $c_{r+1}(\mathfrak{v})=c_r(\mathfrak{t})$. As the regular representation is a faithful, this implies the first formula. The second formula can be proved similarly. □

4.3. **The integral KLR generators.** In Lemma 4.9 and Lemma 4.12, we have found elements in $\mathcal{H}^\Lambda_n(\mathcal{O})$ which intertwine the residue idempotents $f^{\mathcal{O}}_{\mathbf{i}}$. These intertwiners are not quite the elements that we need, however, because they still depend on t, rather than just on e. To remove this dependence on t we will use Proposition 4.6 to renormalise these elements.

By Lemma 4.4, if $h\in\mathcal{H}^\Lambda_n(\mathcal{O})$ then $h=\sum_{\mathbf{i}\in I^n}hf^{\mathcal{O}}_{\mathbf{i}}$, so that h is completely determined by its projections onto the spaces $\mathcal{H}^\Lambda_n(\mathcal{O})f^{\mathcal{O}}_{\mathbf{i}}$. We use this observation to define analogues of the KLR generators in $\mathcal{H}^\Lambda_n(\mathcal{O})$.

Recall from (4.7) that $M_r=1-L_r+tL_{r+1}$. By Corollary 4.8, if $i_r\ne i_{r+1}+1$ then M_r acts invertibly on $f^{\mathcal{O}}_{\mathbf{i}}\mathcal{H}^\Lambda_n(\mathcal{O})$ so $\frac{1}{M_r}f^{\mathcal{O}}_{\mathbf{i}}$ is a well-defined element of $\mathcal{H}^\Lambda_n(\mathcal{O})$.

As in the introduction, define an embedding $I\hookrightarrow\mathbb{Z}; i\mapsto\hat{\imath}$ by defining $\hat{\imath}$ to be the smallest non-negative integer such that $i=\hat{\imath}+e\mathbb{Z}$, for $i\in I$.

4.14. **Definition.** *Suppose that* $1\le r<n$. *Define elements* $\psi^{\mathcal{O}}_r=\sum_{\mathbf{i}\in I^n}\psi^{\mathcal{O}}_rf^{\mathcal{O}}_{\mathbf{i}}$ *in* $\mathcal{H}^\Lambda_n(\mathcal{O})$ *by*

$$\psi^{\mathcal{O}}_rf^{\mathcal{O}}_{\mathbf{i}}=\begin{cases}(T_r+1)\frac{t^{\hat{\imath}_r}}{M_r}f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r=i_{r+1},\\(T_rL_r-L_rT_r)t^{-\hat{\imath}_r}f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r=i_{r+1}+1,\\(T_rL_r-L_rT_r)\frac{1}{M_r}f^{\mathcal{O}}_{\mathbf{i}}, & \text{otherwise.}\end{cases}$$

If $1\le r\le n$ *then define* $y^{\mathcal{O}}_r=\sum_{\mathbf{i}\in I^n}t^{-\hat{\imath}_r}(L_r-[\hat{\imath}_r])f^{\mathcal{O}}_{\mathbf{i}}$.

The order of the terms in the definition of $\psi^{\mathcal{O}}_r$ matters because M_r does not commute with T_r+1 or with $T_rL_r-L_rT_r$ (see Lemma 4.13), although M_r does commute with $f^{\mathcal{O}}_{\mathbf{i}}$. Notice that $\psi^{\mathcal{O}}_r$ is independent of the choice of seminormal coefficient system because the residue idempotents $f^{\mathcal{O}}_{\mathbf{i}}$ are independent of this choice.

One subtlety of Definition 4.14, which we will pay for later, is that it makes use of the embedding $I\hookrightarrow\mathbb{Z}$ in order to give meaning to expressions like $t^{\pm\hat{\imath}_r}$.

4.15. *Remark.* Unravelling the definitions, the element $\psi_r^{\mathcal{O}}\otimes_{\mathcal{O}}1_K$ is a scalar multiple of the choice of KLR generators for $\mathcal{H}_n^\Lambda(\mathscr{K})$ made by Stroppel and Webster [33, (27)]. Similarly, $y_r^{\mathcal{O}}\otimes_{\mathcal{O}}1_K$ is a multiple of the KLR generator y_r defined by Brundan and Kleshchev [6, (4.21)].

4.16. **Proposition.** *The algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated by the elements*

$$\{\,f_{\mathbf{i}}^{\mathcal{O}}\mid \mathbf{i}\in I^n\,\}\cup\{\,\psi_r^{\mathcal{O}}\mid 1\le r<n\,\}\cup\{\,y_r^{\mathcal{O}}\mid 1\le r\le n\,\}.$$

Proof. Let H be the $\mathcal{O}$-subalgebra of $\mathcal{H}_n^\Lambda(\mathcal{O})$ generated by the elements in the statement of the proposition. We need to show that $H=\mathcal{H}_n^\Lambda(\mathcal{O})$. Directly from the definitions, if $1\le r\le n$ then $L_r=\sum_{\mathbf{i}}(t^{\hat{\imath}_r}y_r^{\mathcal{O}}+[i_r])f_{\mathbf{i}}^{\mathcal{O}}\in H$. Therefore, the Gelfand-Zetlin algebra $\mathscr{L}(\mathcal{O})$ is contained in H. Consequently, $M_r\in H$, for $1\le r<n$. By Definition 2.2, $L_rT_r-T_rL_r=T_r(L_{r+1}-L_r)-1+(1-t)L_{r+1}$. By Corollary 4.8(a), if $i_r\ne i_{r+1}$ then $\frac{1}{L_r-L_{r+1}}f_{\mathbf{i}}^{\mathcal{O}}\in\mathscr{L}(\mathcal{O})\subseteq H$. Therefore, since M_r and $f_{\mathbf{i}}^{\mathcal{O}}$ commute, we can write

$$T_rf_{\mathbf{i}}^{\mathcal{O}}=\begin{cases}\big(t^{-\hat{\imath}_r}\psi_r^{\mathcal{O}}M_r-1\big)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+1},\\ \big(-t^{\hat{\imath}_r}\psi_r^{\mathcal{O}}+1+(t-1)L_{r+1}\big)\frac{1}{L_{r+1}-L_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+1}+1\\ \big(-\psi_r^{\mathcal{O}}M_r+1+(t-1)L_{r+1}\big)\frac{1}{L_{r+1}-L_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.}\end{cases}$$

by Definition 4.14. Hence, $T_r=\sum_{\mathbf{i}}T_rf_{\mathbf{i}}^{\mathcal{O}}\in H$. As $T_1,\dots,T_{n-1},L_1,\dots,L_n$ generate $\mathcal{H}_n^\Lambda(\mathcal{O})$ this implies that $H=\mathcal{H}_n^\Lambda(\mathcal{O})$, completing the proof. ∎

We now use the seminormal form to show that the elements in the statement of Proposition 4.16 satisfy most of the relations of Definition 2.10.

4.17. **Lemma.** *Suppose that $1\le r<n$ and $\mathbf{i}\in I^n$. Then $\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}=f_{\mathbf{j}}^{\mathcal{O}}\psi_r^{\mathcal{O}}$, where $\mathbf{j}=s_r\cdot\mathbf{i}$.*

Proof. By Lemma 4.4 and Proposition 4.6, respectively, M_r and $f_{\mathbf{i}}^{\mathcal{O}}$ both belong to $\mathscr{L}(\mathcal{O})$, which is a commutative algebra. Therefore, $\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}$ and $f_{\mathbf{i}}^{\mathcal{O}}$ commute. If $i_r=i_{r+1}$ then

$$\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}=(T_r+1)\frac{t^{\hat{\imath}_r}}{M_r}f_{\mathbf{i}}^{\mathcal{O}}=(T_r+1)f_{\mathbf{i}}^{\mathcal{O}}\frac{t^{\hat{\imath}_r}}{M_r}f_{\mathbf{i}}^{\mathcal{O}}=f_{\mathbf{i}}^{\mathcal{O}}(T_r+1)\frac{t^{\hat{\imath}_r}}{M_r}f_{\mathbf{i}}^{\mathcal{O}}=f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}},$$

where the third equality comes from Lemma 4.9. The remaining cases follow similarly using Lemma 4.12. ∎

As we will work with right modules we need the right-handed analogue of Definition 4.14. Note that if $i_r\ne i_{r+1}+1$ then $f_{\mathbf{i}}^{\mathcal{O}}\frac{1}{M_r}=\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}\in\mathcal{H}_n^\Lambda(\mathcal{O})$ by Proposition 4.6. Similarly, if $i_r\ne i_{r+1}-1$ then $f_{\mathbf{i}}^{\mathcal{O}}\frac{1}{M_r'}=\frac{1}{M_r'}f_{\mathbf{i}}^{\mathcal{O}}\in\mathcal{H}_n^\Lambda(\mathcal{O})$. It follows that all of the expressions in the next lemma make sense.

4.18. **Lemma.** *Suppose $1\le r<n$ and $\mathbf{i}\in I^n$. Then*

$$f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}=\begin{cases}f_{\mathbf{i}}^{\mathcal{O}}\frac{t^{\hat{\imath}_{r+1}}}{M_r'}(T_r-t), & \textit{if } i_i=i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}(T_rL_r-L_rT_r)t^{-\hat{\imath}_{r+1}}, & \textit{if } i_r=i_{r+1}-1,\\ f_{\mathbf{i}}^{\mathcal{O}}\frac{1}{M_r'}(T_rL_r-L_rT_r), & \textit{otherwise.}\end{cases}$$

Proof. By Lemma 4.17, $f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}=f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{j}}^{\mathcal{O}}$ where $\mathbf{j}=s_r\cdot\mathbf{i}$. Therefore,

$$f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}=\begin{cases}f_{\mathbf{i}}^{\mathcal{O}}(1+T_r)\frac{t^{\hat{\imath}_{r+1}}}{M_r}f_{\mathbf{j}}^{\mathcal{O}}, & \text{if } i_i=i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}(T_rL_r-L_rT_r)t^{-\hat{\imath}_{r+1}}f_{\mathbf{j}}^{\mathcal{O}}, & \text{if } i_r=i_{r+1}-1,\\ f_{\mathbf{i}}^{\mathcal{O}}(T_rL_r-L_rT_r)\frac{1}{M_r}f_{\mathbf{j}}^{\mathcal{O}}, & \text{otherwise.}\end{cases}$$

To complete the proof apply Lemma 4.13. ∎

4.19. **Lemma.** *Suppose that* $\mathbf{i},\mathbf{j}\in I^n$ *and* $1\le r,s\le n$. *Then*

$$\sum_{\mathbf{i}\in I^n} f^{\mathcal{O}}_{\mathbf{i}}=1,\quad f^{\mathcal{O}}_{\mathbf{i}}f^{\mathcal{O}}_{\mathbf{j}}=\delta_{\mathbf{i}\mathbf{j}}f^{\mathcal{O}}_{\mathbf{i}},\quad y^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}}=f^{\mathcal{O}}_{\mathbf{i}}y^{\mathcal{O}}_r \quad\textit{and}\quad y^{\mathcal{O}}_r y^{\mathcal{O}}_s=y^{\mathcal{O}}_s y^{\mathcal{O}}_r.$$

Moreover, if $s\ne r,r+1$ *then* $\psi^{\mathcal{O}}_r y^{\mathcal{O}}_s=y^{\mathcal{O}}_s\psi^{\mathcal{O}}_r$, *for* $1\le r<n$ *and* $1\le s\le n$.

Proof. The elements $f^{\mathcal{O}}_{\mathbf{i}}$, for $\mathbf{i}\in I^n$, form a complete set of pairwise orthogonal idempotents by Lemma 4.4, which gives the first two relations. Since $y_r, f^{\mathcal{O}}_{\mathbf{i}}\in\mathscr{L}(\mathcal{O})$ and $\mathscr{L}(\mathcal{O})$ is a commutative algebra, all of the elements $f^{\mathcal{O}}_{\mathbf{i}}$, $y^{\mathcal{O}}_r$ and $y^{\mathcal{O}}_s$ commute.

Now suppose that $s\ne r,r+1$. Then $y^{\mathcal{O}}_s$ commutes with $\frac{1}{M_r}f^{\mathcal{O}}_{\mathbf{i}}$ and with T_r. Hence, $\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}}y^{\mathcal{O}}_s=y^{\mathcal{O}}_s\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}}$, for any $\mathbf{i}\in I^n$. Therefore, $\psi^{\mathcal{O}}_r y^{\mathcal{O}}_s=y^{\mathcal{O}}_s\psi^{\mathcal{O}}_r$. □

4.20. **Lemma.** *Suppose that* $\mathbf{i}\in I^n$. *Then*

$$\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}} (y^{\mathcal{O}}_1-[\kappa_l-\hat{\imath}_1])f^{\mathcal{O}}_{\mathbf{i}}=0.$$

Proof. By Definition 2.2, $\prod_{l=1}^{\ell}(L_1-[\kappa_l])=0$ so that $\prod_{l=1}^{\ell}(L_1-[\kappa_l])f^{\mathcal{O}}_{\mathbf{i}}=0$, for all $\mathbf{i}\in I$. If $\kappa_l\not\equiv i_1 \pmod e$ then $[\hat{\imath}_1]\ne[\kappa_l]$ so that $(L_1-[\kappa_l])$ acts invertibly on $f^{\mathcal{O}}_{\mathbf{i}}\mathcal{H}^{\Lambda}_n$ by Proposition 4.6. Consequently, by Definition 4.14,

$$0=\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}}(t^{\hat{\imath}_1}y^{\mathcal{O}}_1+[\hat{\imath}_1]-[\kappa_l])f^{\mathcal{O}}_{\mathbf{i}}=t^{\hat{\imath}_1\langle\Lambda,\alpha_{i_1}\rangle}\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}}(y^{\mathcal{O}}_1-[\kappa_l-\hat{\imath}_1])f^{\mathcal{O}}_{\mathbf{i}}.$$

As t is invertible in $\mathcal{O}$, the lemma follows. □

Suppose that $\mathfrak{s}$ is a standard tableau, $\mathbf{i}=\operatorname{res}(\mathfrak{s})\in I^n$ and $1\le r<n$. Define

$$\beta_r(\mathfrak{s})=\begin{cases}\dfrac{t^{\hat{\imath}_r-c_r(\mathfrak{s})}\alpha_r(\mathfrak{s})}{[1-\rho_r(\mathfrak{s})]}, & \text{if } i_r=i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_r}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})], & \text{if } i_r=i_{r+1}+1,\\ \dfrac{t^{-\rho_r(\mathfrak{s})}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})]}{[1-\rho_r(\mathfrak{s})]}, & \text{otherwise,}\end{cases} \tag{4.21}$$

and

$$\widehat{\beta}_r(\mathfrak{s})=\begin{cases}\dfrac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{s})}\alpha_r(\mathfrak{s})}{[1+\rho_r(\mathfrak{s})]}, & \text{if } i_r=i_{r+1},\\ -t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})], & \text{if } i_r=i_{r+1}-1,\\ -\dfrac{\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})]}{[1+\rho_r(\mathfrak{s})]}, & \text{otherwise.}\end{cases} \tag{4.22}$$

These scalars describe the action of $\psi^{\mathcal{O}}_r$ and $y^{\mathcal{O}}_r$ upon the seminormal basis.

4.23. **Lemma.** *Suppose that* $1\le r<n$ *and that* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^{\Lambda}_n)$. *Set* $\mathbf{i}=\operatorname{res}(\mathfrak{s})$, $\mathbf{j}=\operatorname{res}(\mathfrak{t})$, $\mathfrak{u}=\mathfrak{s}(r,r+1)$ *and* $\mathfrak{v}=\mathfrak{t}(r,r+1)$. *Then*

$$\psi^{\mathcal{O}}_r f_{\mathfrak{s}\mathfrak{t}}=\beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}}-\delta_{i_ri_{r+1}}\frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}},$$

and

$$f_{\mathfrak{s}\mathfrak{t}}\psi^{\mathcal{O}}_r=\widehat{\beta}_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}}-\delta_{j_rj_{r+1}}\frac{t^{\hat{\jmath}_{r+1}-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}}.$$

Similarly, $y^{\mathcal{O}}_r f_{\mathfrak{s}\mathfrak{t}}=[c_r(\mathfrak{s})-\hat{\imath}_r]f_{\mathfrak{s}\mathfrak{t}}$, *and* $f_{\mathfrak{s}\mathfrak{t}}y^{\mathcal{O}}_r=[c_r(\mathfrak{t})-\hat{\jmath}_r]f_{\mathfrak{s}\mathfrak{t}}$, *for* $1\le r\le n$.

Proof. Applying Definition 4.14 and (3.15),

$$y_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = t^{-\hat{\imath}_r}([c_r(\mathfrak{s})] - [\hat{\imath}_r]) f_{\mathfrak{s}\mathfrak{t}} = [c_r(\mathfrak{s}) - \hat{\imath}_r] f_{\mathfrak{s}\mathfrak{t}}.$$

The proof that $f_{\mathfrak{s}\mathfrak{t}} y_r^{\mathcal{O}} = [c_r(\mathfrak{t}) - \hat{\jmath}_r] f_{\mathfrak{s}\mathfrak{t}}$ is similar. We now consider $\psi_r^{\mathcal{O}}$.

By (3.16), if $\mathbf{k} \in I^n$ then $f_{\mathbf{k}}^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \delta_{\mathbf{i}\mathbf{k}} f_{\mathfrak{s}\mathfrak{t}}$. We use this observation below without mention. By Lemma 4.11, $(T_r L_r - L_r T_r) f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s}) t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})] f_{\mathfrak{u}\mathfrak{t}}$. Hence, $\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}}$ when $i_r \neq i_{r+1}$ by Definition 4.14 and (4.7). Now suppose that $i_r = i_{r+1}$. Then, using (4.7) and (3.15),

$$\begin{aligned}
\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} &= (1 + T_r)\frac{t^{\hat{\imath}_r}}{M_r} f_{\mathfrak{s}\mathfrak{t}} = \frac{t^{\hat{\imath}_r - c_r(\mathfrak{s})}}{[1 - \rho_r(\mathfrak{s})]}\Big(\alpha_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} + \big(1 - \frac{1}{[\rho_r(\mathfrak{s})]}\big) f_{\mathfrak{s}\mathfrak{t}}\Big)\\
&= \beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} - \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]} f_{\mathfrak{s}\mathfrak{t}},
\end{aligned}$$

as required. The formula for $f_{\mathfrak{s}\mathfrak{t}}\psi_r^{\mathcal{O}}$ is proved similarly using Lemma 4.18 in place of Definition 4.14. □

Note that, in general, $\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} \neq (f_{\mathfrak{t}\mathfrak{s}}\psi_r^{\mathcal{O}})^*$.

The next relation can also be proved using Lemma 4.13 and Lemma 4.18.

4.24. **Corollary.** *Suppose that $|r - t| > 1$, for $1 \le r, t < n$. Then $\psi_r^{\mathcal{O}}\psi_t^{\mathcal{O}} = \psi_t^{\mathcal{O}}\psi_r^{\mathcal{O}}$.*

Proof. It follows easily from Lemma 4.23 that $\psi_r\psi_t f_{\mathfrak{s}\mathfrak{t}} = \psi_t\psi_r f_{\mathfrak{s}\mathfrak{t}}$, for all $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Hence, by Lemma 4.4, $\psi_r^{\mathcal{O}}\psi_t^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \psi_t^{\mathcal{O}}\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$, for all $\mathbf{i} \in I^n$. □

4.25. **Lemma.** *Suppose that $1 \le r < n$ and $\mathbf{i} \in I^n$. Then*

$$\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}} \quad \textit{and} \quad y_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = (\psi_r^{\mathcal{O}} y_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}.$$

Proof. Both formulas can be proved similarly, so we consider only the first one. We prove the stronger result that $\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathfrak{s}\mathfrak{t}}$, whenever $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $\operatorname{res}(\mathfrak{s}) = \mathbf{i}$. By (4.3) this implies the lemma.

Suppose first that $i_r = i_{r+1}$. Then, using Lemma 4.23,

$$\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\Big(\beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} - \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]} f_{\mathfrak{s}\mathfrak{t}}\Big).$$

On the other hand, by Lemma 4.23 and (4.21),

$$\begin{aligned}
(y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + 1) f_{\mathfrak{s}\mathfrak{t}} &= [c_r(\mathfrak{u}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} + \Big(1 - \frac{t^{\hat{\imath}_r - c_{r+1}(\mathfrak{s})}[c_r(\mathfrak{s}) - \hat{\imath}_r]}{[\rho_r(\mathfrak{s})]}\Big) f_{\mathfrak{s}\mathfrak{t}}\\
&= [c_r(\mathfrak{u}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} + \frac{[\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})]}{[\rho_r(\mathfrak{s})]} f_{\mathfrak{s}\mathfrak{t}}.
\end{aligned}$$

Therefore, $\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + 1) f_{\mathfrak{s}\mathfrak{t}}$ since $c_r(\mathfrak{u}) = c_{r+1}(\mathfrak{s})$ and $i_r = i_{r+1}$.

If $i_r \neq i_{r+1}$ then the calculation is easier because

$$\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} = y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}},$$

where, for the last equality, we again use the fact that $c_r(\mathfrak{u}) = c_{r+1}(\mathfrak{s})$. □

The following simple combinatorial identity largely determines both the quadratic and the (deformed) braid relations for the $\psi_r^{\mathcal{O}}$, for $1 \le r < n$. This result can be viewed as a graded analogue of the defining property (3.10) of a seminormal coefficient system.

4.26. **Lemma.** *Suppose that* $1 \le r < n$ *and* $\mathfrak{s}, \mathfrak{u} \in \operatorname{Std}(\boldsymbol{\lambda})$ *with* $\mathfrak{u} = \mathfrak{s}(r, r+1)$ *and* $\operatorname{res}(\mathfrak{s}) = \mathbf{i} \in I^n$, *for* $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. *Then*

$$\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}) = \begin{cases} t^{c_r(\mathfrak{s})+c_{r+1}(\mathfrak{s})-\hat{\imath}_r-\hat{\imath}_{r+1}}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})], & \text{if } i_r \leftrightarrows i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})], & \text{if } i_r \to i_{r+1},\\ t^{c_r(\mathfrak{s})-\hat{\imath}_r}[1-\rho_r(\mathfrak{s})], & \text{if } i_r \leftarrow i_{r+1},\\ -\frac{t^{2\hat{\imath}_r-2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2}, & \text{if } i_r = i_{r+1},\\ 1, & \text{otherwise.} \end{cases}$$

Proof. The lemma follows directly from the definition of $\beta_r(\mathfrak{s})$ using (3.10). $\square$

It is time to pay the price for the failure of the embedding $I \hookrightarrow \mathbb{Z}$ to extend to an embedding of quivers. Together with the cyclotomic relation, this is place where the KLR grading fails to lift to the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$. Recall from Definition 4.14 that $y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = t^{-\hat{\imath}_r}(L_r - [\hat{\imath}_r])f_{\mathbf{i}}^{\mathcal{O}}$, where $1 \le r \le n$ and $\mathbf{i} \in I^n$. For $d \in \mathbb{Z}$ define

$$y_r^{\langle d\rangle} f_{\mathbf{i}}^{\mathcal{O}} = t^{d-\hat{\imath}_r}(L_r - [\hat{\imath}_r - d])f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^{\mathcal{O}} + [d])f_{\mathbf{i}}^{\mathcal{O}}. \tag{4.27}$$

In particular, $y_r^{\langle 0\rangle} = y_r^{\mathcal{O}}$ and $y_r^{\langle d\rangle} \otimes_{\mathcal{O}} 1_K = y_r^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K$ whenever e divides $d \in \mathbb{Z}$,

As a final piece of notation, set $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1} \in \mathbb{Z}$, for $\mathbf{i} \in I^n$ and $1 \le r < n$.

4.28. **Proposition.** *Suppose that* $1 \le r < n$ *and* $\mathbf{i} \in I^n$. *Then*

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \to i_{r+1},\\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

Proof. Once again, by (4.3) it is enough to prove the corresponding formulas for $(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}}$, where $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $\mathbf{i} = \operatorname{res}(\mathbf{i})$.

Suppose that $i_r = i_{r+1}$. Let $\mathfrak{u} = \mathfrak{s}(r, r+1)$ and $\mathbf{j} = \operatorname{res}(\mathfrak{u})$. By Lemma 4.23,

$$(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} = \Big(\frac{t^{2\hat{\imath}_r-2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2} + \beta_r(\mathfrak{s})\beta_r(\mathfrak{u})\Big)f_{\mathfrak{st}} - \Big(\frac{\beta_r(\mathfrak{s})t^{\hat{\imath}_r-c_r(\mathfrak{s})}}{[\rho_r(\mathfrak{u})]} + \frac{\beta_r(\mathfrak{s})t^{\hat{\jmath}_r-c_r(\mathfrak{u})}}{[\rho_r(\mathfrak{s})]}\Big)f_{\mathfrak{ut}}.$$

Note that $\rho_r(\mathfrak{s}) = -\rho_r(\mathfrak{u})$ and $i_r = j_r$, so that $t^{\hat{\jmath}_r-c_r(\mathfrak{u})}[\rho_r(\mathfrak{u})] = -t^{\hat{\imath}_r-c_r(\mathfrak{s})}[\rho_r(\mathfrak{s})]$. Hence, using Lemma 4.26, $(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} = 0$ when $i_r = i_{r+1}$ as claimed.

Now suppose that $i_r \ne i_{r+1}$. Then, by Lemma 4.23 and Lemma 4.26,

$$\begin{aligned}(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} &= \beta_r(\mathfrak{s})\beta_r(\mathfrak{u})f_{\mathfrak{st}}\\ &= \begin{cases} t^{c_r(\mathfrak{s})+c_{r+1}(\mathfrak{s})-\hat{\imath}_r-\hat{\imath}_{r+1}}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \to i_{r+1},\\ t^{c_r(\mathfrak{s})-\hat{\imath}_r}[1-\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \leftarrow i_{r+1},\\ f_{\mathfrak{st}}, & \text{otherwise.} \end{cases}\end{aligned}$$

As in Lemma 4.23, if $d \in \mathbb{Z}$ then $y_r^{\langle d\rangle} f_{\mathfrak{st}} = [c_r(\mathfrak{s}) - \hat{\imath}_r + d]f_{\mathfrak{st}}$. So, if $i_r \to i_{r+1}$ then

$$\begin{aligned}(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})f_{\mathfrak{st}} &= \big([c_r(\mathfrak{s}) + 1 - \hat{\imath}_{r+1}] - [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\big)f_{\mathfrak{st}}\\ &= t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}} = (\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}}.\end{aligned}$$

The cases when $i_r \to i_{r+1}$ and $i_r \leftrightarrows i_{r+1}$ are similar. $\square$

Set $\mathcal{B}_r^{\mathcal{O}} = \psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}$, for $1 \le r < n-1$.

4.29. **Proposition.** *Suppose that $1 \le r < n$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, with $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ for $\mathbf{i} \in I^n$. Then*

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ -t^{1+\rho_r(\mathbf{i})} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \to i_{r+1}, \\ f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We mimic the proof of the braid relations from Lemma 3.13.

Define (not necessarily standard) tableaux $\mathfrak{u}_1 = \mathfrak{s}(r, r+1)$, $\mathfrak{u}_2 = \mathfrak{s}(r+1, r+2)$, $\mathfrak{u}_{12} = \mathfrak{u}_1(r+1, r+2)$, $\mathfrak{u}_{21} = \mathfrak{u}_2(r, r+1)$ and $\mathfrak{u}_{121} = \mathfrak{u}_{12}(r, r+1) = \mathfrak{u}_{212}$. To ease notation set $i = i_r$, $j = i_{r+1}$ and $k = i_{r+2}$. The relationship between these tableaux, and their residues $\{\operatorname{res}_s(\mathfrak{u}) \mid r \le s \le r+2\} = \{i, j, k\}$, is illustrated in the following diagram.

$$\begin{tikzcd}[column sep=small]
 & & \mathfrak{s} \sim (i,j,k) \arrow[dll, "s_r"'] \arrow[drr, "s_{r+1}"] & & \\
\mathfrak{u}_1 \sim (j,i,k) \arrow[d, "s_{r+1}"'] & & & & \mathfrak{u}_2 \sim (i,k,j) \arrow[d, "s_r"] \\
\mathfrak{u}_{12} \sim (j,k,i) \arrow[drr, "s_r"'] & & & & \mathfrak{u}_{21} \sim (k,i,j) \arrow[dll, "s_{r+1}"] \\
 & & \mathfrak{u}_{121} = \mathfrak{u}_{212} \sim (k,j,i) & &
\end{tikzcd}$$

Note that if any tableau $\mathfrak{u} \in \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_{12}.\mathfrak{u}_{21}, \mathfrak{u}_{121}\}$ is not standard then, by definition, $f_{\mathfrak{u}\mathfrak{t}} = 0$ so this term can be ignored in all of the calculations below.

We need to compute $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$. To start with, observe that by Lemma 4.23, the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is equal to

$$\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) - \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21}).$$

By definition, the scalars $[\rho_r(\mathfrak{s})]$ and $[1-\rho_r(\mathfrak{s})]$ are determined by the positions of r and $r+1$ in $\mathfrak{s}$, so it is easy to see that

$$\begin{aligned} \rho_r(\mathfrak{s}) &= \rho_{r+1}(\mathfrak{u}_{21}), & \rho_r(\mathfrak{u}_1) &= \rho_{r+1}(\mathfrak{u}_{121}), & \rho_r(\mathfrak{u}_2) &= \rho_{r+1}(\mathfrak{u}_1), \\ \rho_r(\mathfrak{u}_{12}) &= \rho_{r+1}(\mathfrak{s}), & \rho_r(\mathfrak{u}_{21}) &= \rho_{r+1}(\mathfrak{u}_{12}), & \rho_r(\mathfrak{u}_{121}) &= \rho_{r+1}(\mathfrak{u}_2). \end{aligned} \tag{4.30}$$

Observe that $\alpha_r(\mathfrak{s})\alpha_{r+1}(\mathfrak{u}_1)\alpha_r(\mathfrak{u}_{12}) = \alpha_{r+1}(\mathfrak{s})\alpha_r(\mathfrak{u}_2)\alpha_{r+1}(u_{21})$ by (3.9). Keeping track of the exponent of t, (4.21) and (4.30) now imply that $\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) = \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21})$. Note that (3.9) is crucial here. Therefore, the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is zero for any choice of i, j and k. As the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is always zero we will omit $f_{\mathfrak{u}_{121}\mathfrak{t}}$ from most of the calculations which follow.

There are five cases to consider.

Case 1. *i, j and k are pairwise distinct.*
By Lemma 4.23 and the last paragraph,

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \big(\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) - \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21})\big) f_{\mathfrak{u}_{121}\mathfrak{t}} = 0,$$

as required by the statement of the proposition.

Case 2. *$i = j \ne k$.*
In this case, using Lemma 4.23,

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \Big(- \frac{t^{\hat{\imath} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]} \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2) + \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2) \frac{t^{\hat{\imath} - c_{r+2}(\mathfrak{u}_{21})}}{[\rho_{r+1}(\mathfrak{u}_{21})]} \Big) f_{\mathfrak{u}_{21}\mathfrak{t}}.$$

Now $c = c_{r+1}(\mathfrak{u}_{21})$ and $c_{r+1}(\mathfrak{s}) = c_{r+2}(\mathfrak{u}_{21})$, as in (4.30). Hence, $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = 0$ when $i = j \neq k$.

Case 3. $i \neq j = k$.
This is almost identical to Case 2, so we leave the details to the reader.

Case 4. $i = k \neq j$.
Typographically, it is convenient to set $c = c_r(\mathfrak{s})$, $c' = c_{r+1}(\mathfrak{s})$ and $c'' = c_{r+2}(\mathfrak{s})$. According to the statement of the proposition, this is the only case where $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} \neq 0$. Using Lemma 4.23, we see that

$$\begin{aligned}
\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} &= \Big(-\beta_r(\mathfrak{s}) \frac{t^{\hat{\imath}-c_{r+2}(\mathfrak{u}_1)}}{[\rho_{r+1}(\mathfrak{u}_1)]} \beta_r(\mathfrak{u}_1) + \beta_{r+1}(\mathfrak{s}) \frac{t^{\hat{\imath}-c_{r+1}(\mathfrak{u}_2)}}{[\rho_r(\mathfrak{u}_2)]} \beta_{r+1}(\mathfrak{u}_2) \Big) f_{\mathfrak{s}\mathfrak{t}}. \\
&= \frac{t^{\hat{\imath}-c''}}{[c-c'']} \big(-\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}_1) + \beta_{r+1}(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_2) \big) f_{\mathfrak{s}\mathfrak{t}}.
\end{aligned}$$

Expanding the last equation using Lemma 4.26 shows that

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \begin{cases}
-\dfrac{t^{c}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})] - t^{c''}[1-\rho_{r+1}(\mathfrak{s})][1+\rho_{r+1}(\mathfrak{s})]}{t^{c''-c'+\hat{\jmath}}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \leftrightarrows j, \\
-\dfrac{[1+\rho_r(\mathfrak{s})] - [1-\rho_{r+1}(\mathfrak{s})]}{t^{c''-\hat{\imath}-c'+\hat{\jmath}}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \to j, \\
-\dfrac{t^{c}[1-\rho_r(\mathfrak{s})] - t^{c''}[1+\rho_{r+1}(\mathfrak{s})]}{t^{c''}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \leftarrow j, \\
0, & \text{otherwise.}
\end{cases}$$

(Note that, by assumption, the case $i = j$ does not arise.) If $i \leftrightarrows j$ then a straightforward calculation shows that in this case

$$\begin{aligned}
\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} &= -\Big([c'-\hat{\jmath}+2] + [c'-\hat{\jmath}] - [c+1-\hat{\jmath}] - [c''+1-\hat{\jmath}] \Big) f_{\mathfrak{s}\mathfrak{t}} \\
&= -\big(y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} \big) f_{\mathfrak{s}\mathfrak{t}},
\end{aligned}$$

where the last equality uses Lemma 4.25 and the observation that, because $e = 2$, we have $\{1 \pm \rho_r(\mathbf{i})\} = \{0, 2\}$ and $\{\hat{\imath}, \hat{\jmath}\} = \{0, 1\}$. A similar, but easier, calculation shows that if $i \to j$ then $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = -t^{1+\hat{\imath}-\hat{\jmath}} f_{\mathfrak{s}\mathfrak{t}} = -t^{1+\rho_r(\mathbf{i})} f_{\mathfrak{s}\mathfrak{t}}$ and if $i \leftarrow j$ then $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{s}\mathfrak{t}}$. If $i \neq j$ and $i \not- j$ then we have already seen that $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = 0$, so this completes the proof of Case 4.

Case 5. $i = j = k$.
We continue to use the notation for c, c', c'' from Case 4. By Lemma 4.23 (compare with the proof of Lemma 3.13), $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is equal to

$$\begin{aligned}
-\Big(\tfrac{t^{3\hat{\imath}-2c'-c''}}{[\rho_r(\mathfrak{s})]^2[\rho_{r+1}(\mathfrak{s})]} - \tfrac{t^{3\hat{\imath}-c'-2c''}}{[\rho_{r+1}(\mathfrak{s})]^2[\rho_r(\mathfrak{s})]} + \tfrac{t^{\hat{\imath}-c''}\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}_1)}{[\rho_{r+1}(\mathfrak{u}_1)]} - \tfrac{t^{\hat{\imath}-c''}\beta_{r+1}(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_2)}{[\rho_r(\mathfrak{u}_2)]} \Big) f_{\mathfrak{s}\mathfrak{t}} \\
+t^{2\hat{\imath}}\beta_r(\mathfrak{s}) \Big(\tfrac{t^{-c''-c}}{[\rho_{r+1}(\mathfrak{u}_1)][\rho_r(\mathfrak{u}_1)]} + \tfrac{t^{-c'-c''}}{[\rho_r(\mathfrak{s})][\rho_{r+1}(\mathfrak{s})]} - \tfrac{t^{-2c''}}{[\rho_{r+1}(\mathfrak{s})][\rho_{r+1}(\mathfrak{u}_1)]} \Big) f_{\mathfrak{u}_1\mathfrak{t}} \\
+t^{2\hat{\imath}}\beta_{r+1}(\mathfrak{s}) \Big(\tfrac{t^{-c'-c''}}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{u}_2)]} - \tfrac{t^{-c''-c'}}{[\rho_r(\mathfrak{u}_2)][\rho_{r+1}(\mathfrak{u}_2)]} - \tfrac{t^{-c''-c'}}{[\rho_{r+1}(\mathfrak{s})][\rho_r(\mathfrak{s})]} \Big) f_{\mathfrak{u}_2\mathfrak{t}} \\
-t^{\hat{\imath}-c''}\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1) \Big(\tfrac{1}{\rho_r(\mathfrak{u}_{12})} - \tfrac{1}{[\rho_{r+1}(\mathfrak{s})]} \Big) f_{\mathfrak{u}_{12}\mathfrak{t}} \\
-t^{\hat{\imath}-c'}\beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2) \Big(\tfrac{1}{[\rho_r(\mathfrak{s})]} - \tfrac{1}{[\rho_{r+1}(\mathfrak{u}_{21})]} \Big) f_{\mathfrak{u}_{21}\mathfrak{t}}.
\end{aligned}$$

Using (4.30) it is easy to see that the coefficients of $f_{\mathfrak{u}_{12}\mathfrak{t}}$ and $f_{\mathfrak{u}_{21}\mathfrak{t}}$ are both zero. On the other hand, if $t \neq 1$ then the coefficient of $t^{2\hat{\imath}}\beta_r(\mathfrak{s}) f_{\mathfrak{u}_1\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is

$$\frac{t-1}{(t^{c'}-t^{c})(t^{c}-t^{c''})} + \frac{t-1}{(t^{c}-t^{c'})(t^{c'}-t^{c''})} - \frac{t-1}{(t^{c'}-t^{c''})(t^{c}-t^{c''})},$$

which is easily seen to be zero. The case when $t = 1$ now follows by specialisation. Similarly, the coefficient of $f_{\mathfrak{u}_2\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is also zero. Finally, using Lemma 4.26 and (4.30), the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is zero as the four terms above, which

give the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in the displayed equation, cancel out in pairs. Hence, $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = 0$ when $i = j = k$, as required.

This completes the proof. □

We need one more relation, which is a deformation of Corollary 2.16.

4.31. **Lemma.** *Suppose that $2 \le r \le n$ and $\mathbf{i} \in I^n$. Then*

$$\prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (y_r^{\mathcal{O}} - [c_r(\mathfrak{t}) - \hat{\imath}_r]) f_{\mathbf{i}}^{\mathcal{O}} = 0.$$

Proof. This is an immediate consequence of (4.3) and Lemma 4.23. □

4.4. **A deformation of the quiver Hecke algebra.** Using the results of the last two sections we now describe $\mathcal{H}_n^\Lambda(\mathcal{O})$ by generators and relations using the '$\mathcal{O}$-KLR generators' of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

Suppose that $(\mathcal{O}, t)$ is an idempotent subring of $\mathscr{K}$. So far we have not used the assumption that $[de] \in J(\mathcal{O})$, for $d \in \mathbb{Z}$. This comes into play in the next theorem.

Note that the relations for $y_2^{\mathcal{O}}, \dots y_n^{\mathcal{O}}$ in the next theorem are not quite the same as those in Theorem A from the introduction.

4.32. **Theorem.** *Suppose that $(\mathcal{O}, t)$ is an e-idempotent subring of $\mathscr{K}$. Then the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n \,\} \cup \{\, \psi_r^{\mathcal{O}} \mid 1 \le r < n \,\} \cup \{\, y_r^{\mathcal{O}} \mid 1 \le r \le n \,\}$$

subject only to the following relations:

$$\prod_{\substack{1\le l\le \ell\\ \kappa_l \equiv i_1 \ (mod\ e)}} (y_1^{\mathcal{O}} - [\kappa_l - \hat{\imath}_1]) f_{\mathbf{i}}^{\mathcal{O}} = 0 = \prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (y_r^{\mathcal{O}} - [c_r(\mathfrak{t}) - \hat{\imath}_r]) f_{\mathbf{i}}^{\mathcal{O}}, \quad \textit{for } 2 \le r < n,$$

$$f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{i}\mathbf{j}} f_{\mathbf{i}}^{\mathcal{O}}, \qquad \textstyle\sum_{\mathbf{i}\in I^n} f_{\mathbf{i}}^{\mathcal{O}} = 1, \qquad y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} y_r^{\mathcal{O}},$$

$$\begin{aligned}
\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= f_{s_r\cdot\mathbf{i}}^{\mathcal{O}} \psi_r^{\mathcal{O}}, & y_r^{\mathcal{O}} y_s^{\mathcal{O}} &= y_s^{\mathcal{O}} y_r^{\mathcal{O}},\\
\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= (y_r^{\mathcal{O}} \psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, & y_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= (\psi_r^{\mathcal{O}} y_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}},\\
\psi_r^{\mathcal{O}} y_s^{\mathcal{O}} &= y_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, & &\textit{if } s \ne r, r+1,\\
\psi_r^{\mathcal{O}} \psi_s^{\mathcal{O}} &= \psi_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, & &\textit{if } |r-s| > 1,
\end{aligned}$$

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftrightarrows i_{r+1},\\
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \to i_{r+1},\\
(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftarrow i_{r+1},\\
0, & \textit{if } i_r = i_{r+1},\\
f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise},
\end{cases}$$

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\
-t^{1+\rho_r(\mathbf{i})} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \to i_{r+1},\\
f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \leftarrow i_{r+1},\\
0, & \textit{otherwise},
\end{cases}$$

where we set $y_r^{\langle d\rangle} = t^d y_r^{\mathcal{O}} + [d]$, for $d \in \mathbb{Z}$.

Proof. Let $R_n(\mathcal{O})$ be the abstract algebra defined by the generators and relations in the statement of the theorem. By the results in the last two sections, the elements given in Definition 4.14 satisfy all of the relations of the corresponding generators of $R_n(\mathcal{O})$. Hence, by Proposition 4.16, there is a surjective $\mathcal{O}$-algebra homomorphism $\theta{:}R_n(\mathcal{O}) \twoheadrightarrow \mathcal{H}_n^\Lambda(\mathcal{O})$, which maps the generators of $R_n(\mathcal{O})$ to the corresponding elements of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

If $w \in \mathfrak{S}_n$ then set $\psi^{\mathcal{O}}_w = \psi^{\mathcal{O}}_{r_1} \dots \psi^{\mathcal{O}}_{r_k}$, where $w = s_{r_1} \dots s_{r_k}$ is a reduced expression for w. In general, $\psi^{\mathcal{O}}_w$ will depend upon the choice of reduced expression, however, using the relations in $R_n(\mathcal{O})$ it follows that every element in $R_n(\mathcal{O})$ can be written as a linear combination of elements of the form $f(y)\psi_w e(\mathbf{i})$, where $f(y) \in \mathcal{O}[y^{\mathcal{O}}_1, \dots, y^{\mathcal{O}}_n]$, $w \in \mathfrak{S}_n$ and $\mathbf{i} \in I^n$. Hence, because of the first two relations in the statement of the theorem, $R_n(\mathcal{O})$ is finitely generated as an $\mathcal{O}$-module.

Now suppose that $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$ and let $K = \mathcal{O}/\mathfrak{m} \cong \mathcal{O}_{\mathfrak{m}}/\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$ and $\zeta = t + \mathfrak{m}$. Then $1 + \zeta + \dots + \zeta^{e-1} = 0$ in K, since $[e] \in J(\mathcal{O}) \subseteq \mathfrak{m}$. Note also that $1 + \zeta + \dots + \zeta^{k-1} \neq 0$ if $k \notin e\mathbb{Z}$ since $\mathcal{O}$ is an e-idempotent subring. Consequently, $y^{\langle de\rangle}_r \otimes 1_K = y^{\mathcal{O}}_r \otimes 1_K$, for all $d \in \mathbb{Z}$. It is easy to see that all of the shifts $1 \pm \rho_r(\mathbf{i})$ appearing in the statement of theorem are equal to either 0 or to e. Therefore, in view of Corollary 2.16, upon base change to K the relations of $R_n(\mathcal{O}_{\mathfrak{m}}) \otimes_{\mathcal{O}_{\mathfrak{m}}} K$ coincide with the relations of the quiver Hecke algebra $\mathcal{R}^\Lambda_n(K)$, see Definition 2.10 and Theorem 2.14. Consequently, $R_n(\mathcal{O}_{\mathfrak{m}}) \otimes_{\mathcal{O}_{\mathfrak{m}}} K \cong \mathcal{R}^\Lambda_n(K)$, so that $\dim R_n(\mathcal{O}_{\mathfrak{m}}) \otimes_{\mathcal{O}_{\mathfrak{m}}} K = \dim \mathcal{H}^\Lambda_n(K)$ by Theorem 2.14.

By the last paragraph, if $K = \mathcal{O}/\mathfrak{m}$, for any maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, then $\dim R_n(\mathcal{O}_{\mathfrak{m}}) \otimes_{\mathcal{O}_{\mathfrak{m}}} K = \dim \mathcal{H}^\Lambda_n(K) = \ell^n n!$. Moreover, by the second paragraph of the proof, $R_n(\mathcal{O}_{\mathfrak{m}})$ is a finitely generated $\mathcal{O}_{\mathfrak{m}}$-algebra. Therefore, Nakayama's lemma applies and it implies that $R_n(\mathcal{O}_{\mathfrak{m}})$ is a free $\mathcal{O}_{\mathfrak{m}}$-module of rank $\ell^n n!$. Hence, the map $\theta_{\mathfrak{m}} : R_n(\mathcal{O}_{\mathfrak{m}}) \xrightarrow{\sim} \mathcal{H}^\Lambda_n(\mathcal{O}_{\mathfrak{m}})$ is an isomorphism of $\mathcal{O}_{\mathfrak{m}}$-algebras. It follows that θ is an isomorphism of $\mathcal{O}$-algebras, as required. □

4.33. *Remarks.* (a) All of the relations in Theorem 4.32 are deformations of the relations in Definition 2.10 except for the relations

$$\prod_{\mathfrak{t}\in \mathrm{Std}(\mathbf{i})} (y^{\mathcal{O}}_r - [c_r(\mathfrak{t}) - \hat{\imath}_r]) f^{\mathcal{O}}_{\mathbf{i}} = 0,$$

for $2 \le r \le n$. These relations are needed to ensure that $R_n(\mathcal{O})$, as defined in the proof of Theorem 4.32, is finitely generated as an $\mathcal{O}$-algebra. This is crucial to the proof of Theorem 4.32 because without this we cannot apply Nakayama's Lemma (and hence Theorem 2.14). It should be possible to prove Theorem 4.32 directly, without appealing to Nakayama's Lemma and Theorem 2.14, by adapting the arguments of [6, Theorem 3.3].

(b) In proving Theorem 2.14, Brundan and Kleshchev [6] construct a family of isomorphisms $\mathcal{R}^\Lambda_n \xrightarrow{\sim} \mathcal{H}^\Lambda_n(\mathscr{K})$ that depend on a choice of polynomials $Q_r(\mathbf{i})$ which can be varied subject to certain constraints. In our setting this amounts to choosing certain invertible 'scalars' $q_r(\mathbf{i})$, which are rational functions in L_r and L_{r+1}, and defining

$$\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = \begin{cases} (T_r + 1)\frac{t^{\hat{\imath}_r}}{M_r} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r = i_{r+1},\\ (T_r L_r - L_r T_r) q_r(\mathbf{i}) f^{\mathcal{O}}_{\mathbf{i}}, & \text{otherwise,} \end{cases}$$

such that the corresponding β-coefficients still satisfy the constraints of Lemma 4.26. To make this explicit, if $i_r \neq i_{r+1}$ and $\mathfrak{s} \in \mathrm{Std}(\mathbf{i})$ then Lemma 4.23 becomes

$$\psi^{\mathcal{O}}_r f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})] q_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} - \delta_{i_r i_{r+1}} \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]},$$

where $q_r(\mathfrak{s}) \in \mathscr{K}$ is the scalar such that $q_r(\mathbf{i}) f_{\mathfrak{s}\mathfrak{t}} = q_r(\mathfrak{s}) f_{\mathfrak{s}\mathfrak{t}}$ and where $\mathfrak{u} = \mathfrak{s}(r, r+1)$. Therefore, in order for Lemma 4.26 to hold we require that

$$q_r(\mathfrak{s}) q_r(\mathfrak{u}) = \begin{cases} t^{-\hat{\imath}_r - \hat{\imath}_{r+1}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ \frac{t^{-\hat{\imath}_{r+1}}}{[1-\rho_r(\mathfrak{s})]}, & \text{if } i_r \rightarrow i_{r+1},\\ \frac{t^{-\hat{\imath}_r}}{[1-\rho_r(\mathfrak{s})]}, & \text{if } i_r \leftarrow i_{r+1},\\ \frac{1}{[1-\rho_r(\mathfrak{s})][\rho_r(\mathfrak{u})]}, & \text{if } i_r \not- i_{r+1}, \end{cases}$$

and that these scalars satisfy a "braid relation" as in (3.9). If the $q_r(\mathbf{i})$ satisfy these two identities then it is easy to see that argument used to prove Theorem 4.32 applies, virtually without change, using these more general elements. The key point is that Lemma 4.26 still holds. The corresponding identities in Brundan and Kleshchev's work are [6, (3.28), (3.29), (4.34) and (4.35)].

We end this section by using Theorem 4.32 to give an upper bound for the nilpotency index of the KLR generators $y_1, \dots, y_n$. As in the introduction, if $1 \le r \le n$ and $\mathbf{i} \in I^n$ set

$$\mathscr{E}_r(\mathbf{i}) = \{\, c_r(\mathfrak{t}) - \hat{\imath}_r \mid \mathfrak{t} \in \operatorname{Std}(\mathbf{i}) \,\}$$

and define $E_r(\mathbf{i}) = \#\mathscr{E}_r(\mathbf{i})$. For example, $\mathscr{E}_1(\mathbf{i}) \subseteq \{\kappa_1 - \hat{\imath}_1, \dots, \kappa_\ell - \hat{\imath}_1\}$ and $E_1(\mathbf{i}) = (\Lambda, \alpha_{i_1})$. In general, $\mathscr{E}_r(\mathbf{i}) \subseteq \{\, ke \mid k \in \mathbb{Z} \,\}$ since $c_r(\mathfrak{t}) \equiv i_r \pmod{e}$ if $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$.

4.34. **Proposition.** *Suppose that $1 \le r \le n$ and $\mathbf{i} \in I^n$. Then*

$$\prod_{c \in \mathscr{E}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0.$$

Proof. By Lemma 4.4 and Lemma 4.23,

$$\begin{aligned}\prod_{c \in \mathscr{E}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} &= \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \prod_{c \in \mathscr{E}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}} \\ &= \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}} \prod_{c \in \mathscr{E}_r(\mathbf{i})} ([c_r(\mathfrak{t}) - \hat{\imath}_r] - [c]) f_{\mathfrak{t}\mathfrak{t}} = 0,\end{aligned}$$

where the last equality follows because $c_r(\mathfrak{t}) - \hat{\imath}_r \in \mathscr{E}_r(\mathbf{i})$, for all $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. □

Even though Proposition 4.34 is very easy to prove within our framework, it has several very interesting consequences. The first is that because $\mathcal{H}_n^\Lambda(\mathcal{O}) \cong R_n(\mathcal{O})$, where we use the notation from the proof of Theorem 4.32, we can improve upon the presentation of $\mathcal{H}_n^\Lambda(\mathcal{O})$ given by Theorem 4.32 and so prove Theorem A from the introduction.

4.35. **Corollary.** *Suppose that $(\mathcal{O}, t)$ is an e-idempotent subring of $\mathscr{K}$. Then, as an $\mathcal{O}$-algebra, $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated by the elements $\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n \,\} \cup \{\, \psi_r^{\mathcal{O}} \mid 1 \le r < n \,\} \cup \{\, y_r^{\mathcal{O}} \mid 1 \le r \le n \,\}$ subject only to the relations in Theorem A.*

Secondly, we obtain the corresponding result for the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$. Note that, in general, $E_r(\mathbf{i}) \le N_r(\mathbf{i}) = \#\operatorname{Std}(\mathbf{i})$, so the next result improves upon Corollary 2.16.

4.36. **Corollary.** *Suppose that $\mathbf{i} \in I^n$ and $1 \le r \le n$. Then $y_r^{E_r(\mathbf{i})} e(\mathbf{i}) = 0$ in $\mathcal{R}_n^\Lambda$.*

When $e = 0$ Brundan and Kleshchev [6, Conjecture 2.3] conjectured that $y_r^\ell = 0$, for $1 \le r \le n$. Hoffnung and Lauda proved this conjecture as the main result in their paper [13]. Using Corollary 4.36 we obtain a quick proof of this result and, at the same time, a generalization of it to include the case when $e \ge n$.

4.37. **Corollary.** *Suppose that $e = 0$ or $e \ge n$. Then $y_r^\ell = 0$, for $1 \le r \le n$.*

Proof. If $e = 0$ then we may assume that $e \gg 0$ by Corollary 2.15. Consequently, it is enough to consider the case when $e \ge n$. By definition, if $1 \le l \le \ell$ then a node $\alpha = (a, b, l) \in \boldsymbol{\lambda}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$, has residue i_r if and only if $i_r = b - a + \kappa_l \pmod{e}$. Since $e \ge n$, and $|a - b| < n$, it follows that the content $b - a + \kappa_l$ of α is uniquely determined by i_r and l. That is, all of the nodes of residue i_r in a given component of any multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ have the same content. Hence, $E_r(\mathbf{i}) \le \ell$. As $y_r^\ell = \sum_{\mathbf{i}} y_r^\ell e(\mathbf{i})$ the result is now a consequence of Corollary 4.36. □

5. Integral bases for $\mathcal{H}_n^\Lambda(\mathcal{O})$

Now that we have proved Theorem A, we begin to use the machinery of seminormal forms to study the cyclotomic quiver Hecke algebras $\mathcal{R}_n^\Lambda$. In this chapter we reconstruct the 'natural' homogeneous bases for the cyclotomic Hecke algebras $\mathcal{H}_n^\Lambda(K)$ and their Specht modules over a field.

5.1. **The ψ-basis.** Theorem 4.32 links the KLR grading on $\mathcal{H}_n^\Lambda \cong \mathcal{R}_n^\Lambda$ with the semisimple representation theory of $\mathcal{H}_n^\Lambda(\mathscr{K})$. We next want to try and understand the graded Specht modules of $\mathcal{H}_n^\Lambda$ [8, 15, 22] in terms of the seminormal form. We start by lifting the homogeneous basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_n^\Lambda$ to $\mathcal{H}_n^\Lambda(\mathcal{O})$. This turns out to be easier than the approach taken in [15]. Throughout this section, $\mathcal{O}$ is an e-idempotent subring of $\mathscr{K}$.

By Theorem 4.32, there is a unique anti-isomorphism $\star$ of $\mathcal{H}_n^\Lambda(\mathcal{O})$ such that

$$(\psi_r^{\mathcal{O}})^\star = \psi_r^{\mathcal{O}}, \quad (y_s^{\mathcal{O}})^\star = y_s^{\mathcal{O}} \quad \text{and} \quad (f_{\mathbf{i}}^{\mathcal{O}})^\star = f_{\mathbf{i}}^{\mathcal{O}},$$

for $1 \le r < n$, $1 \le s \le n$ and $\mathbf{i} \in I^n$. Lemma 4.23 shows that, in general, the automorphisms $*$ and $\star$ do not coincide.

Recall from Definition 3.7 that a $\star$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of two-sided eigenvalues for $\mathscr{L}$ such that $f_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{t}\mathfrak{s}}^\star$, for all $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. We define a $\star$-seminormal coefficient system to be a set of scalars $\{\beta_r(\mathfrak{t})\}$ which satisfy the identity in Lemma 4.26 and the "braid relations" of (3.9) (with α replaced by β).

The main difference between a $*$-seminormal basis and a $\star$-seminormal basis is that $T_r f_{\mathfrak{s}\mathfrak{t}} = (f_{\mathfrak{t}\mathfrak{s}} T_r)^*$ for a $*$-seminormal basis whereas $\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = (f_{\mathfrak{t}\mathfrak{s}} \psi_r^{\mathcal{O}})^\star$ for a $\star$-seminormal basis.

5.1. **Lemma.** *Suppose that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a $\star$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. Then there exists a unique $\star$-seminormal coefficient system $\{\beta_r(\mathfrak{t})\}$ such that if $1 \le r < n$ and $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ then*

$$f_{\mathfrak{s}\mathfrak{t}}\psi_r^{\mathcal{O}} = \beta_r(\mathfrak{v}) f_{\mathfrak{s}\mathfrak{v}} - \delta_{i_r i_{r+1}} \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}},$$

where $\mathfrak{v} = \mathfrak{t}(r, r+1)$ and $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$, for $\mathbf{i} \in I^n$. Conversely, as in Theorem 3.14, a $\star$-seminormal coefficient system, together with a choice of scalars $\{\, \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$, determines a unique $\star$-seminormal basis.

Proof. By (4.21), a set of scalars $\{\beta_r(\mathfrak{t})\}$ is a $\star$-seminormal coefficient system if and only if $\{\alpha_r(\mathfrak{t})\}$ is a $*$-seminormal coefficient system, where

$$\alpha_r(\mathfrak{t}) = \begin{cases} \beta_r(\mathfrak{t}) t^{c_r(\mathfrak{t}) - \hat{\imath}_r}[1 - \rho_r(\mathfrak{t})], & \text{if } i_r = i_{r+1}, \\ \dfrac{\beta_r(\mathfrak{t}) t^{\hat{\imath}_r - c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]}, & \text{if } i_r = i_{r+1} + 1, \\ \dfrac{\beta_r(\mathfrak{t})[1 - \rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{t})]}, & \text{otherwise.} \end{cases}$$

Therefore, as seminormal coefficient systems are determined by the action of the corresponding generators of $\mathcal{H}_n^\Lambda$ on its right regular representation, the result follows from Theorem 3.14 and Lemma 4.23. $\square$

Henceforth, we will work with $\star$-seminormal bases.

Exactly as in Theorem 3.14, if $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a $\star$-seminormal basis then there exists scalars $\gamma_{\mathfrak{t}} \in \mathscr{K}$ such that $f_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{u}\mathfrak{t}} \gamma_{\mathfrak{t}} f_{\mathfrak{s}\mathfrak{v}}$, for $(\mathfrak{s},\mathfrak{t}), (\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Repeating the argument of Corollary 3.17, these scalars satisfy the following recurrence relation.

5.2. **Corollary.** *Suppose that* $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ *and that* $\mathfrak{v} = \mathfrak{t}(r, r+1)$ *is standard, where* $1 \le r < n$. *Then* $\beta_r(\mathfrak{v})\gamma_{\mathfrak{t}} = \beta_r(\mathfrak{t})\gamma_{\mathfrak{v}}$.

Motivated by [15], we now define a new basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ which is cellular with respect to the involution $\star$. Fix $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and let $\mathbf{i}^{\boldsymbol{\lambda}} = (i_1^{\boldsymbol{\lambda}}, \dots, i_n^{\boldsymbol{\lambda}})$, so that $i_r^{\boldsymbol{\lambda}} = \operatorname{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r)$ for $1 \le r \le n$. Following [15, Definition 4.7], define

$$\mathscr{A}_{\boldsymbol{\lambda}}(r) = \left\{ \alpha \;\middle|\; \begin{array}{c} \alpha \text{ is an addable } i_r^{\boldsymbol{\lambda}}\text{-node of the multipartition} \\ \operatorname{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow r}) \text{ which is } \textit{below } (\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(r) \end{array} \right\},$$

for $1 \le r \le n$.

Up until now we have worked with an arbitrary seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. In order to define a 'nice' basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ which is compatible with Theorem 4.32 we now fix the choice of γ-coefficients by requiring that

$$\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = \prod_{r=1}^{n} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} [c_\alpha - c_r(\mathfrak{t}^{\boldsymbol{\lambda}})], \tag{5.3}$$

for all $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Together with a choice of seminormal coefficient system, this determines $\gamma_{\mathfrak{t}}$ for all $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ by Corollary 5.2. By definition, $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ is typically a non-invertible element of $\mathcal{O}$. Nonetheless, if $\mathbf{i} \in I^n$ then $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{H}_n^\Lambda(\mathcal{O})$ by Lemma 4.4.

We also fix a choice of seminormal coefficient system by requiring that $\beta_r(\mathfrak{s}) = 1$ whenever $\mathfrak{s} \rhd \mathfrak{t} = \mathfrak{s}(r, r+1)$, for $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$. More precisely, if $\mathbf{i} \in I$ and $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ then we define

$$\beta_r(\mathfrak{s}) = \begin{cases} 1, & \text{if } \mathfrak{s} \rhd \mathfrak{t} \text{ or } i_r \not\!\!- i_{r+1}, \\ -\frac{t^{2\hat{\imath}_r - 2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2}, & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r = i_{r+1}, \\ t^{c_r(\mathfrak{s}) + c_{r+1}(\mathfrak{s}) - \hat{\imath}_r - \hat{\imath}_{r+1}}[1 - \rho_r(\mathfrak{s})][1 + \rho_r(s)], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \leftrightarrows i_{r+1}, \\ t^{c_r(\mathfrak{s}) - \hat{\imath}_r}[1 - \rho_r(\mathfrak{s})], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \leftarrow i_{r+1}, \\ t^{c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}}[1 + \rho_r(s)], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \rightarrow i_{r+1}. \end{cases} \tag{5.4}$$

where $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathfrak{t} = \mathfrak{s}(r, r+1)$ is standard, for $1 \le r < n$. The reader is invited to check that this defines a $\star$-seminormal coefficient system. As the definition of $\psi_r^{\mathcal{O}}$ is independent of the choice of seminormal coefficient system this choice is not strictly necessary for what follows but it simplifies many of the formulas.

By Lemma 5.1, this choice of $\star$-seminormal coefficient system and γ-coefficients determines a unique $\star$-seminormal basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_n^\Lambda(\mathscr{K})$. We will use this basis to define new homogeneous basis of $\mathcal{H}_n^\Lambda$. The first step is to define

$$\begin{aligned} y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} &= \prod_{r=1}^{n} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^{\boldsymbol{\lambda}})}([c_\alpha] - L_r) f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} \\ &= \prod_{r=1}^{n} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{\hat{\imath}_r^{\boldsymbol{\lambda}} - c_r(\mathfrak{t}^{\boldsymbol{\lambda}})}([c_\alpha - \hat{\imath}_r^{\boldsymbol{\lambda}}] - y_r^{\mathcal{O}}) f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}}, \end{aligned}$$

where the second equation follows by rewriting $L_k f_{\mathbf{i}}^{\mathcal{O}}$ in terms of $y_k f_{\mathbf{i}}^{\mathcal{O}}$ as in the proof of Proposition 4.16. In particular, these equations show that $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K$ is a monomial in $y_1, \dots, y_n$ and, further, that it is (up to a sign) equal to the element $y^{\boldsymbol{\lambda}}$ defined in [15, Definition 4.15].

The next result is a essentially a translation of [15, Lemma 4.13] into the current setting for the special case of the tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$.

5.5. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then there exist scalars $a_{\mathfrak{s}} \in \mathscr{K}$ such that*

$$y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} + \sum_{\mathfrak{s} \blacktriangleright \mathfrak{t}^{\boldsymbol{\lambda}}} a_{\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}.$$

In particular, $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}}$ is a non-zero element of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

Proof. By Lemma 4.4, $f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} = \sum_{\mathfrak{s}} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}$, so that $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} = \sum_{\mathfrak{s} \in \mathrm{Std}(\mathbf{i}^{\boldsymbol{\lambda}})} a_{\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}$, for some $a_{\mathfrak{s}} \in \mathscr{K}$, by (3.15). It remains to show that $a_{\mathfrak{t}^{\boldsymbol{\lambda}}} = 1$ and that $a_{\mathfrak{s}} \neq 0$ only if $\mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}$. Using (3.15), and recalling the definition of $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ from (5.3),

$$\frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}} y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}} \prod_{r=1}^{n} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^{\boldsymbol{\lambda}})}([c_\alpha] - [c_r(\mathfrak{t}^{\boldsymbol{\lambda}})]) \cdot f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}.$$

To complete the proof we claim that there exist scalars $a_{\mathfrak{s}}(k) \in \mathscr{K}$, $1 \le k \le n$, such that

$$\prod_{r=1}^{k} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^{\boldsymbol{\lambda}})}([c_\alpha] - L_r) f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\mathbf{i}^{\boldsymbol{\lambda}}) \\ \mathfrak{s}_{\downarrow k} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow k}}} a_{\mathfrak{s}}(k) f_{\mathfrak{s}\mathfrak{s}}$$

where $a_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k) = 1$. We prove this by induction on k. If $k = 1$ then the result is immediate from (3.15). Suppose that $k > 1$. By induction, it is enough to show that

$$([c_\alpha] - L_k) f_{\mathfrak{s}\mathfrak{s}} = ([c_\alpha] - [c_k(\mathfrak{s})]) f_{\mathfrak{s}\mathfrak{s}} = 0$$

whenever $\mathfrak{s}_{\downarrow(k-1)} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow(k-1)}$ and $\mathfrak{s}_{\downarrow k} \not\trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow k}$, for $\mathfrak{s} \in \mathrm{Std}(\mathbf{i}^{\boldsymbol{\lambda}})$. Fix such a tableau $\mathfrak{s}$. Since $\mathfrak{s}_{\downarrow(k-1)} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow(k-1)}$ we must have $(\mathfrak{s}_{\downarrow k})^{(l)} = \emptyset$ whenever $l > \mathrm{comp}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k)$, so the node $\alpha = \mathfrak{s}^{-1}(k)$ must be below $(\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(k)$. Therefore, $\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(k)$, and $c_k(\mathfrak{s}) = \mathrm{c}_\alpha$ for this α, and forcing $a_{\mathfrak{s}}(k) = 0$ as claimed. This completes the proof. $\square$

For each $w \in \mathfrak{S}_n$ we now fix a reduced expression $w = s_{r_1} \dots s_{r_k}$ for w, with $1 \le r_j < n$ for $1 \le j \le k$, and define $\psi_w^{\mathcal{O}} = \psi_{r_1}^{\mathcal{O}} \dots \psi_{r_k}^{\mathcal{O}}$. By Theorem 4.32 the elements $\psi_r^{\mathcal{O}}$ do not satisfy the braid relations so, in general, $\psi_w^{\mathcal{O}}$ will depend upon this (fixed) choice of reduced expression.

5.6. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Define*

$$\psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\star y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^{\boldsymbol{\lambda}}}^{\mathcal{O}} \psi_{d(\mathfrak{t})}^{\mathcal{O}}.$$

for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$.

We can now lift the graded cellular basis of [15, Definitions 5.1] to $\mathcal{H}_n^\Lambda(\mathcal{O})$.

5.7. **Theorem.** *Suppose that $\mathcal{O}$ is an idempotent subring. Then*

$$\{\, \psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu}) \text{ for } \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda \,\}$$

is a cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ with respect to the involution $\star$.

Proof. In view of (3.15) and Lemma 4.23, Lemma 5.5 implies that

$$\psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})} a_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}}, \tag{5.8}$$

for some $a_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$. Therefore, $\{\, \psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ is a basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. In fact, these elements are a basis for $\mathcal{H}_n^\Lambda(\mathcal{O})$ because if $h \in \mathcal{H}_n^\Lambda(\mathcal{O})$ then we can write $h = \sum r_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$. Pick $(\mathfrak{s}, \mathfrak{t})$ to be minimal with respect to dominance such that $r_{\mathfrak{s}\mathfrak{t}} \neq 0$. Then $r_{\mathfrak{s}\mathfrak{t}} \in \mathcal{O}$ because $h \in \mathcal{H}_n^\Lambda(\mathcal{O})$. Consequently, $h - r_{\mathfrak{s}\mathfrak{t}} \psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ so, by continuing in this way, we can write h as a linear combination of the ψ-basis.

It remains to show that the ψ-basis is cellular with respect to the involution $\star$. By definition, if $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ then $y_{\mathcal{O}}^{\boldsymbol{\lambda}}$ and $f_{\mathbf{i}^\lambda}^{\mathcal{O}}$ commute and they are fixed by the automorphism $\star$. Therefore, $(\psi_{\mathfrak{st}}^{\mathcal{O}})^\star = \psi_{\mathfrak{ts}}^{\mathcal{O}}$, for all $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. By Lemma 5.1, the $\star$-seminormal basis $\{f_{\mathfrak{st}}\}$ is a cellular basis with cellular involution $\star$. It remains to verify (GC$_2$) from Definition 2.4. As in Theorem 3.14, the seminormal basis $\{f_{\mathfrak{uv}}\}$ is cellular. Therefore, if $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\boldsymbol{\lambda})$ and $h \in \mathcal{H}_n^\Lambda(\mathcal{O})$ then, using (5.8) twice,

$$\psi_{\mathfrak{st}}^{\mathcal{O}} h = (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\star \psi_{\mathfrak{t}^\lambda\mathfrak{t}}^{\mathcal{O}} \equiv (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\star \Big(f_{\mathfrak{t}^\lambda\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} a_{\mathfrak{v}} f_{\mathfrak{t}^\lambda a}\Big) h \equiv (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\star \sum_{\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})} a'_{\mathfrak{v}} f_{\mathfrak{t}^\lambda a}$$

$$\equiv (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\star \sum_{\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda}} b_{\mathfrak{v}} \psi_{\mathfrak{t}^\lambda\mathfrak{v}}^{\mathcal{O}} \quad \equiv \sum_{\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda}} b_{\mathfrak{v}} \psi_{\mathfrak{sv}}^{\mathcal{O}} \quad (\mathrm{mod}\ \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}),$$

where $a_{\mathfrak{v}}, a'_{\mathfrak{v}} \in \mathscr{K}$ and $b_{\mathfrak{v}} \in \mathcal{O}$ with the scalars $b_{\mathfrak{v}}$ being independent of $\mathfrak{s}$. Hence, (GC$_2$) holds, completing the proof. □

If $K = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ then $\mathcal{H}_n^\Lambda(K) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$. Set $\psi_{\mathfrak{st}} = \psi_{\mathfrak{st}}^{\mathcal{O}} \otimes 1_K$.

5.9. **Corollary** ([15, Theorem 5.8]). *Suppose that $K = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}$. Then $\{\, \psi_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathcal{P}_n^\Lambda \,\}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda(K)$ with $\deg \psi_{\mathfrak{st}} = \deg \mathfrak{s} + \deg \mathfrak{t}$, for $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$.*

By (5.8) the basis elements in $\{\psi_{\mathfrak{st}}\}$ are scalar multiples of the basis elements constructed in [15, Theorem 5.8].

5.2. **Graded Specht modules and Gram determinants.** By Theorem 5.7, $\{\psi_{\mathfrak{st}}^{\mathcal{O}}\}$ is a cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ so we can use it to define Specht modules for $\mathcal{H}_n^\Lambda(\mathcal{O})$ which specialise to the graded Specht modules in characteristic zero and in positive characteristic.

5.10. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. The Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is the right $\mathcal{H}_n^\Lambda(\mathcal{O})$-module with basis $\{\, \psi_{\mathfrak{t}}^{\mathcal{O}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$, where $\psi_{\mathfrak{t}}^{\mathcal{O}} = \psi_{\mathfrak{t}^\lambda\mathfrak{t}}^{\mathcal{O}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}(\mathcal{O})$.*

By Theorem 5.7 and [15, Corollary 5.10], ignoring the grading, $S^{\boldsymbol{\lambda}}(\mathcal{O}) \otimes_{\mathcal{O}} K$ can be identified with the graded Specht module $S^{\boldsymbol{\lambda}}$ of $\mathcal{H}_n^\Lambda$ defined by Brundan, Kleshchev and Wang [8]. The action of $\mathcal{H}_n^\Lambda(\mathscr{K})$ on a graded Specht module is completely determined by the relations for these modules which are given in [22]. In contrast, in view of (5.8) and Theorem 4.32, the action of $\mathcal{H}_n^\Lambda(\mathcal{O})$ on the Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is completely determined by the (choice of) seminormal form.

We now turn to computing the determinant of the Gram matrix

$$\mathcal{G}^{\boldsymbol{\lambda}} = \big(\langle \psi_{\mathfrak{s}}^{\mathcal{O}}, \psi_{\mathfrak{t}}^{\mathcal{O}} \rangle\big)_{\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})}.$$

A priori, it is unclear how the bilinear form on $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is related to the usual (ungraded) bilinear from on the Specht module which is defined using the Murphy basis which we considered in Theorem 3.22. The main problem in relating these two bilinear forms is that the cellular algebra involutions $*$ and $\star$, which are used to define these bilinear forms, are different.

Note that the cellular algebra involutions $*$ and $\star$ on $\mathcal{H}_n^\Lambda(\mathcal{O})$ naturally extend to involutions on the algebra $\mathcal{H}_n^\Lambda(\mathscr{K})$. The key point to understanding the graded bilinear form is the following.

5.11. **Lemma.** *Suppose that $\mathfrak{t} \in \mathrm{Std}(n)$. Then $(F_{\mathfrak{t}})^\star = F_{\mathfrak{t}}$.*

Proof. By definition, $F_{\mathfrak{t}}$ is a linear combination of products of Jucys-Murphy elements, so it can also be written as a polynomial, with coefficients in $\mathscr{K}$, in $y_r^{\mathcal{O}}$, $f_{\mathbf{i}}^{\mathcal{O}}$, for $1 \le r \le n$ and $\mathbf{i} \in I^n$. As $(y_r^{\mathcal{O}})^\star = y_r^{\mathcal{O}}$, $(f_{\mathbf{i}}^{\mathcal{O}})^\star = f_{\mathbf{i}}^{\mathcal{O}}$, for $1 \le r \le n$ and $\mathbf{i} \in I^n$, the result follows. □

Recall that if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\psi^{\mathcal{O}}_{\mathfrak{t}} = \psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$ is a basis element of the Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$. In order to compute $\det \mathcal{G}^{\boldsymbol{\lambda}}$, set $f_{\mathfrak{t}} = \psi^{\mathcal{O}}_{\mathfrak{t}} F_{\mathfrak{t}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Recall that $S^{\boldsymbol{\lambda}}(\mathscr{K}) = S^{\boldsymbol{\lambda}}(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$.

5.12. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_n$. Then $\{\, f_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$ is a basis of $S^{\boldsymbol{\lambda}}(\mathscr{K})$. Moreover, $\det \mathcal{G}^{\boldsymbol{\lambda}} = \det\big(\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle\big) = \prod_{\mathfrak{s}\in\operatorname{Std}(\boldsymbol{\lambda})} \gamma_{\mathfrak{s}}$.*

Proof. By definition, $f_{\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + (\mathcal{H}^{\Lambda}_n(\mathscr{K}))^{\rhd\boldsymbol{\lambda}}$. Therefore, $f_{\mathfrak{t}} \in S^{\boldsymbol{\lambda}}(\mathscr{K})$ and $f_{\mathfrak{t}} = \psi^{\mathcal{O}}_{\mathfrak{t}} + \sum_{\mathfrak{v}\rhd\mathfrak{t}} r_{\mathfrak{t}\mathfrak{v}}\psi^{\mathcal{O}}_{\mathfrak{v}}$ by (5.8), for some scalars $r_{\mathfrak{t}\mathfrak{v}} \in \mathscr{K}$. Set $r_{\mathfrak{t}\mathfrak{t}} = 1$ and $U = \big(r_{\mathfrak{t}\mathfrak{v}}\big)$. Then $\{\, f_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$ is a $\mathscr{K}$-basis of $S^{\boldsymbol{\lambda}}(\mathscr{K})$ and $\mathcal{G}^{\boldsymbol{\lambda}} = (U^{-1})^{tr}\big(\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle\big)U^{-1}$ Taking determinants shows that $\det \mathcal{G}^{\boldsymbol{\lambda}} = \deg\big(\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle\big)$ since U is unitriangular. To complete the proof observe that $\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \equiv f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} f_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} = \delta_{\mathfrak{s}\mathfrak{t}}\gamma_{\mathfrak{s}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, where we are implicitly using Lemma 5.11. The result follows. □

Lemma 5.12 is subtly different from (3.19) because, in spite of our notation, the $\gamma_{\mathfrak{t}}$'s appearing in the two formulas satisfy different recurrence relations.

5.13. **Lemma.** *Suppose that $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_n$. Then $\gamma_{\mathfrak{t}} = u_{\mathfrak{t}}\Phi_e(t)^{\deg_e(\mathfrak{t})}$, for some unit $u_{\mathfrak{t}} \in \mathcal{O}^{\times}$.*

Proof. We argue by induction on the dominance order on $\operatorname{Std}(\boldsymbol{\lambda})$. If $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ then (5.3) ensures that $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = u_{\mathfrak{t}^{\boldsymbol{\lambda}}}\Phi_e(t)^{\deg_e(\mathfrak{t}^{\boldsymbol{\lambda}})}$, for some unit $u_{\mathfrak{t}^{\boldsymbol{\lambda}}} \in \mathcal{O}$. Now suppose that $\mathfrak{t}^{\boldsymbol{\lambda}} \rhd \mathfrak{t}$. Then there exists a standard tableau $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ such that $\mathfrak{s} \rhd \mathfrak{t}$ and $\mathfrak{t} = \mathfrak{s}(r, r+1)$, where $1 \le r < n$. Arguing exactly as in Corollary 5.2 shows that $\beta_r(\mathfrak{s})\gamma_{\mathfrak{t}} = \beta_r(\mathfrak{t})\gamma_{\mathfrak{s}}$. Therefore, $\gamma_{\mathfrak{t}} = \frac{\beta_r(\mathfrak{t})}{\beta_r(\mathfrak{s})}\gamma_{\mathfrak{s}} = \beta_r(\mathfrak{t})\gamma_{\mathfrak{s}}$. Hence, the lemma follows by induction exactly as in the proof of Theorem 3.22. □

5.14. *Remark.* Looking at the definition of a $\star$-seminormal coefficient system shows that the quantities $\frac{\beta_r(\mathfrak{t})}{\beta_r(\mathfrak{s})}$, which are used in the proof of Lemma 5.13, are independent of the choice of $\star$-seminormal coefficient system. This shows that the choice of $\star$-seminormal coefficient system made in (5.4) really is only for convenience.

By general nonsense, the determinants of $\mathcal{G}^{\boldsymbol{\lambda}}$ and $\underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ differ by a scalar in $\mathscr{K}$. The last two results readily imply the next theorem, the real content of which is that this scalar is a unit in $\mathcal{O}$.

5.15. **Theorem.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_n$. Then $\det \mathcal{G}^{\boldsymbol{\lambda}} = u\Phi_e(t)^{\deg_e(\boldsymbol{\lambda})}$, for some unit $u \in \mathcal{O}^{\times}$. Consequently, $\det \mathcal{G}^{\boldsymbol{\lambda}} = u' \det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$, for some unit $u' \in \mathcal{O}^{\times}$.*

If $\mathbf{i} \in I^n$ and $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_n$ let $\operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda}) = \{\, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \operatorname{res}(\mathfrak{t}) = \mathbf{i}\,\}$.

The Specht module $S^{\boldsymbol{\lambda}}$ over $\mathcal{O}$ decomposes as a direct sum of generalised eigenspaces as an $\mathscr{L}(\mathcal{O})$-module: $S^{\boldsymbol{\lambda}} = \bigoplus_{\mathbf{i}\in I^n} S^{\boldsymbol{\lambda}}_{\mathbf{i}}$, where $S^{\boldsymbol{\lambda}}_{\mathbf{i}} = S^{\boldsymbol{\lambda}} f^{\mathcal{O}}_{\mathbf{i}}$. The weight space $S^{\boldsymbol{\lambda}}_{\mathbf{i}}$ has basis $\{\, \psi^{\mathcal{O}}_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda})\,\}$ and the bilinear linear form $\langle\ ,\ \rangle$ on $S^{\boldsymbol{\lambda}}$ respects the weight space decomposition of $S^{\boldsymbol{\lambda}}$. Set

$$\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda})} \deg \mathfrak{t}.$$

and let $\mathcal{G}^{\boldsymbol{\lambda}}_{\mathbf{i}}$ be restriction of the Gram matrix of $S^{\boldsymbol{\lambda}}$ to $S^{\boldsymbol{\lambda}}_{\mathbf{i}}$, for $\mathbf{i} \in I^n$. Then we have the following refinement of Theorem 5.15.

5.16. **Corollary.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_n$ and $\mathbf{i} \in I^n$. Then $\deg \mathcal{G}^{\boldsymbol{\lambda}}_{\mathbf{i}} = u_{\mathbf{i}}\Phi_e(t)^{\deg_{e,\mathbf{i}}(\boldsymbol{\lambda})}$, for some unit $u_{\mathbf{i}} \in \mathcal{O}^{\times}$. Moreover, $\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) \ge 0$.*

6. A distinguished homogeneous basis for $\mathcal{H}_n^\Lambda$

The ψ-basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$, the homogeneous bases of $\mathcal{H}_n^\Lambda$ constructed in [15], and the homogeneous basis of the graded Specht modules given by Brundan, Kleshchev and Wang [8], are all indexed by pairs of standard tableaux. Unfortunately, unlike in the ungraded case, these basis elements depend upon choices of reduced expressions for the permutations corresponding to these tableaux. In this section we construct new bases for these modules which depend only on the corresponding tableaux.

6.1. **A new basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$.** To construct our new basis for $\mathcal{H}_n^\Lambda$ we need to work over a complete discrete valuation ring. We start by setting up the necessary machinery.

Recall that the algebra $\mathcal{H}_n^\Lambda$ is defined over the field K with parameter ξ and that $e > 1$ is minimal such that $[e]_\xi = 0$. Let x be an indeterminate over K and let $\mathcal{O} = K[x]_{(x)}$ and $t = x+\xi$. Then $(\mathcal{O}, t)$ is an idempotent subring by Example 4.2(b) and $K(x)$ is the field of fractions of $\mathcal{O}$. Note that $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} = x\mathcal{O}$.

Let $\widehat{\mathcal{O}}$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. Then $\widehat{\mathcal{O}}$ is a complete discrete valuation ring with field of fractions $K((x))$ Let $\widehat{\mathscr{K}} = K((x))$ be the $\mathfrak{m}$-adic completion of $K(x)$. Then $\widehat{\mathcal{O}}$ is an idempotent subring of $\widehat{\mathscr{K}}$.

Define a valuation on $\widehat{\mathscr{K}}^\times$ by setting $\nu_x(a) = n$ if $a = ux^n$, where $n \in \mathbb{Z}$ and $u \in \widehat{\mathcal{O}}^\times$ is a unit in $\widehat{\mathcal{O}}$. We need to work with a complete discrete valuation ring because of the following fundamental but elementary fact which is proved, for example, as [32, Proposition II.5].

6.1. **Lemma.** *Suppose that $a \in \widehat{\mathscr{K}}$. Then a can be written uniquely as a convergent series*

$$a = \sum_{n\in\mathbb{Z}} a_n x^n, \qquad \text{with } a_n \in K,$$

such that if $a \neq 0$ then $a_n \neq 0$ only if $n \geq \nu_x(a)$. Moreover, $a \in \widehat{\mathcal{O}}$ if and only if $a_n = 0$ for all $n < 0$.

In particular, $x^{-1}K[x^{-1}] \cap \widehat{\mathcal{O}} = 0$, where we embed $x^{-1}K[x^{-1}]$ into $\widehat{\mathscr{K}}$ in the obvious way.

6.2. **Theorem.** *Suppose that $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. There exists a unique element $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \in \mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$ such that*

$$B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)\\ (\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})}} p_{\mathfrak{u}\mathfrak{v}}^{\mathfrak{s}\mathfrak{t}}(x^{-1}) f_{\mathfrak{u}\mathfrak{v}},$$

where $p_{\mathfrak{u}\mathfrak{v}}^{\mathfrak{s}\mathfrak{t}}(x) \in xK[x]$. Moreover, $\{\, B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ is a cellular basis of $\mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$.

Proof. The existence of an element $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}}$ with the required properties follows directly from (5.8) and Lemma 6.1 using Gaussian elimination. (See the proof of Proposition 6.4, below, which proves a stronger result in characteristic zero.) To prove uniqueness of the element $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}}$, suppose, by way of contradiction, that there exist two elements $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}}$ and $B'_{\mathfrak{s}\mathfrak{t}}$ in $\mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$ with the required properties. Then $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} - B'_{\mathfrak{s}\mathfrak{t}} = \sum r_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}} \in \mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$ and, by assumption, $r_{\mathfrak{u}\mathfrak{v}} \in x^{-1}K[x^{-1}]$ with $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})$. Pick $(\mathfrak{a},\mathfrak{b})$ minimal with respect to dominance such that $r_{\mathfrak{a}\mathfrak{b}} \neq 0$. Then, by Theorem 5.7, if we write $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} - B'_{\mathfrak{s}\mathfrak{t}}$ as a linear combination of ψ-basis elements then $\psi_{\mathfrak{a}\mathfrak{b}}^{\widehat{\mathcal{O}}}$ appears with coefficient $r_{\mathfrak{a}\mathfrak{b}}$. Therefore, $r_{\mathfrak{a}\mathfrak{b}} \in x^{-1}K[x^{-1}] \cap \widehat{\mathcal{O}} = 0$, a contradiction. Hence, $B_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = B'_{\mathfrak{s}\mathfrak{t}}$ as claimed.

By (5.8), the transition matrix between the B-basis and the ψ-basis is unitriangular, so $\{B^{\mathcal{O}}_{\mathfrak{st}}\}$ is a basis of $\mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$. To show that the B-basis is cellular we need to check properties (GC$_1$)–(GC$_3$) from Definition 2.4. We have already verified (GC$_1$) Moreover, (GC$_3$) holds because $(B^{\mathcal{O}}_{\mathfrak{st}})^\star = B^{\mathcal{O}}_{\mathfrak{ts}}$ by the uniqueness statement in Proposition 6.4 since $\{f_{\mathfrak{uv}}\}$ is $\star$-seminormal basis. It remains to prove (GC$_2$), which we do in three steps.

Step 1. We claim that if $h \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} h \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} b_{\mathfrak{v}} B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for some scalars $b_{\mathfrak{v}} \in \widehat{\mathcal{O}}$ which depend only on $\mathfrak{t}$, $\mathfrak{v}$ and h (and not on $\mathfrak{t}^{\boldsymbol{\lambda}}$).

To see this first note that $\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{v}\blacktriangleright\mathfrak{t}} a_{\mathfrak{v}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}}$ by (5.8), for some $a_{\mathfrak{v}} \in K(x)$. Therefore, it follows by induction on the dominance order that if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{v}\blacktriangleright\mathfrak{t}} p_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for some $p_{\mathfrak{tv}} \in x^{-1}K[x^{-1}]$. As the seminormal basis is cellular, and the transition matrix between the seminormal basis and the B-basis is unitriangular, our claim now follows.

Step 2. As the Specht module $S^{\boldsymbol{\lambda}}$ is cyclic there exists an element $D^{\mathcal{O}}_{\mathfrak{t}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ such that $B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} \equiv B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}$. We claim that

$$B^{\mathcal{O}}_{\mathfrak{st}} \equiv (D^{\mathcal{O}}_{s})^\star B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for all $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$.

To prove this claim, embed $\mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ in $\mathcal{H}^\Lambda_n(\widehat{\mathscr{K}})$. Note that $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{uv}} = 0$ if $\mathfrak{u} \ne \mathfrak{t}^{\boldsymbol{\lambda}}$, so we may assume that $D^{\mathcal{O}}_{\mathfrak{t}} \equiv \sum_{\mathfrak{v}} q_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}$, for some $q_{\mathfrak{tv}} \in \widehat{\mathscr{K}}$. Then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} \equiv B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} = \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} q_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}.$$

Therefore, $q_{\mathfrak{tv}} = \frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}} p_{\mathfrak{tv}}$, where $p_{\mathfrak{tv}} \in \delta_{\mathfrak{tv}} + x^{-1}K[x^{-1}]$ is as in Step 1. In particular, $q_{\mathfrak{tt}} = \frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}}$ and $q_{\mathfrak{tv}} \ne 0$ only if $\mathfrak{v} \trianglerighteq \mathfrak{t}$. Consequently,

$$\begin{aligned}(D^{\mathcal{O}}_{s})^\star B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} &\equiv \sum_{\substack{(\mathfrak{u},\mathfrak{v})\,\trianglerighteq\,(\mathfrak{s},\mathfrak{t})\\ \mathfrak{u},\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})}} q_{\mathfrak{su}} q_{\mathfrak{tv}} f_{\mathfrak{ut}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} = \sum_{(\mathfrak{u},\mathfrak{v})\,\trianglerighteq\,(\mathfrak{s},\mathfrak{t})} \gamma^2_{\mathfrak{t}^{\boldsymbol{\lambda}}} q_{\mathfrak{su}} q_{\mathfrak{tv}} f_{\mathfrak{uv}}\\ &= f_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} p_{\mathfrak{su}} p_{\mathfrak{tv}} f_{\mathfrak{uv}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}.\end{aligned}$$

By construction, $(D^{\mathcal{O}}_{s})^\star B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$. Consequently, our claim now follows using the uniqueness property of $B^{\mathcal{O}}_{\mathfrak{st}}$ since $p_{\mathfrak{su}} p_{\mathfrak{tv}} \in x^{-1}K[x^{-1}]$ when $\mathfrak{s} \ne \mathfrak{u}$ or $\mathfrak{t} \ne \mathfrak{v}$.

Step 3. We can now verify (GC$_2$). If $h \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ then, using steps 1 and 2,

$$B^{\mathcal{O}}_{\mathfrak{st}} h \equiv (D^{\mathcal{O}}_{s})^\star B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} h \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} b_{\mathfrak{v}} (D^{\mathcal{O}}_{s})^\star B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} b_{\mathfrak{v}} B^{\mathcal{O}}_{\mathfrak{sv}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

where $b_{\mathfrak{v}}$ depends only on $\mathfrak{t}$, $\mathfrak{v}$ and h and not on $\mathfrak{s}$. Hence, the B-basis satisfies all of the cellular basis axioms and the theorem is proved. □

By Theorem 6.2, if $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}(\mathcal{P}^\Lambda_n)$ then $B^{\mathcal{O}}_{\mathfrak{st}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$, however, our notation suggests that $B^{\mathcal{O}}_{\mathfrak{st}} \in \mathcal{H}^\Lambda_n(\mathcal{O})$, where $\mathcal{O} = K[x]_{(x)}$. The next result justifies our notation and shows that we can always work over the ring $\mathcal{O}$.

6.3. **Corollary.** *Let $\mathcal{O} = K[x]_{(x)}$. Then $\{\, B^{\mathcal{O}}_{\mathfrak{st}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}^\Lambda_n) \,\}$ is a graded cellular basis of $\mathcal{H}^\Lambda_n(\mathcal{O})$.*

Proof. Fix $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Then it is enough to prove that $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$. First note that by construction the $\star$-seminormal basis is defined over the rational function field $K(x)$, so $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}$ is defined over the ring $R = K(x) \cap \widehat{\mathcal{O}}$ since if $(\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1}) \in K[x^{-1}] \subset K(x)$ by Theorem 6.2. Every element of $K(x)$ can be written in the form $f(x)/g(x)$, for $f(x), g(x) \in K[x]$ with $\gcd(f,g) = 1$. Expanding f/g into a power series, as in Lemma 6.1, it is not difficult to see that if $f/g \in \widehat{\mathcal{O}}$ then $g(0) \neq 0$. Therefore, $R \subseteq \mathcal{O}$ so that $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}$ is defined over $\mathcal{O}$ as claimed. □

By similar arguments, $D^{\mathcal{O}}_{\mathfrak{t}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$, for all $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$.

If K is a field of characteristic zero then we can determine the degree of the polynomials $p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}} \neq 0$, for $(\mathfrak{u},\mathfrak{v}) \trianglerighteq (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$.

6.4. **Proposition.** *Suppose that K is a field of characteristic zero. Suppose that $(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})$ for $(\mathfrak{s},\mathfrak{t}), (\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\boldsymbol{\lambda})$. Then $p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x) \in xK[x]$ and*

$$\deg p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x) \leq \tfrac{1}{2}(\deg \mathfrak{u} - \deg \mathfrak{s} + \deg \mathfrak{v} - \deg \mathfrak{t}).$$

In particular, $p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x) \neq 0$ only if $\deg \mathfrak{u} + \deg \mathfrak{v} \geq \deg \mathfrak{s} + \deg \mathfrak{t}$.

Proof. We argue by induction on the dominance orders on $\mathcal{P}_n^\Lambda$ and $\operatorname{Std}(\mathcal{P}_n^\Lambda)$. Note that $\deg p(x) = d$ if and only if $\nu_x\big(p(x^{-1})\big) = -d$. For convenience, throughout the proof given two tableaux $\mathfrak{s},\mathfrak{u} \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ set $\deg(\mathfrak{s},\mathfrak{u}) = \deg \mathfrak{s} - \deg \mathfrak{u}$. Therefore, the proposition is equivalent to the claim that $\nu_x\big(p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}\big) \geq \deg(\mathfrak{s},\mathfrak{u}) + \deg(\mathfrak{t},\mathfrak{v})$.

Suppose first that $\boldsymbol{\lambda} = (n|0|\dots|0)$. Then $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$ and $\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ so there is nothing to prove. Hence, we may assume that that $\boldsymbol{\lambda} \neq (n|0|\dots|0)$ and that the proposition holds for all more dominant shapes.

Next, consider the case when $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$. By the proof of Lemma 5.5, if $\mathfrak{s} \in \operatorname{Std}(\mathbf{i}^{\boldsymbol{\lambda}})$ and $\mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}$ then $y^{\boldsymbol{\lambda}}_{\mathcal{O}} f_{\mathfrak{s}\mathfrak{s}} = u'_{\mathfrak{s}} \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{s}\mathfrak{s}}$ for some unit $u'_{\mathfrak{s}} \in \mathcal{O}^\times$. Therefore, by Lemma 5.13, there exist units $u_{\mathfrak{s}} \in \mathcal{O}^\times$ so that in $\mathcal{H}_n^\Lambda(\mathcal{O})$

$$\psi^{\widehat{\mathcal{O}}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \sum_{\mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}} \frac{u'_{\mathfrak{s}} \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} + \sum_{\mathfrak{s} \blacktriangleright \mathfrak{t}^{\boldsymbol{\lambda}}} u_{\mathfrak{s}} \Phi_e(t)^{\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})} f_{\mathfrak{s}\mathfrak{s}}.$$

Since $t = x + \xi$, the constant term of $\Phi_e(t)$ is $\Phi_e(\xi) = 0$, so x divides $\Phi_e(t)$ and $\nu_x(\Phi_e(t)^{\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})}) = \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})$ since the coefficient of x in $\Phi_e(t)$ is non-zero. (If K is field of positive characteristic this may not be true.) Expanding each unit $u_{\mathfrak{s}}$ into a power series, as in Lemma 6.1, the coefficient of $f_{\mathfrak{s}\mathfrak{s}}$ can be written as $b_{\mathfrak{s}} + c_{\mathfrak{s}}$ where $b_{\mathfrak{s}} \in x^{-1}K[x^{-1}]$ and $c_{\mathfrak{s}} \in \mathcal{O}$. In particular, if $b_{\mathfrak{s}} \neq 0$ and $c_{\mathfrak{s}} \neq 0$ then $\nu_x(c_{\mathfrak{s}}) \geq 0 > \nu_x(b_{\mathfrak{s}})$ and $\nu_x(c_{\mathfrak{s}}) > \nu_x(b_{\mathfrak{s}}) \geq \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})$. Pick $\mathfrak{t}$ minimal with respect to dominance such that $c_{\mathfrak{t}} \neq 0$. Note that $\nu_x(c_{\mathfrak{t}}) \geq \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t})$, with equality only if $b_{\mathfrak{t}} = 0$. Using induction, replace $\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ with the element $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} - c_{\mathfrak{t}} B^{\mathcal{O}}_{\mathfrak{t}\mathfrak{t}}$. By construction $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \in \mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$ and, by (5.8), the coefficient of $f_{\mathfrak{t}\mathfrak{t}}$ in $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ is $b_{\mathfrak{t}} \in x^{-1}K[x^{-1}]$. If $(\mathfrak{u},\mathfrak{v}) \trianglerighteq (\mathfrak{t},\mathfrak{t})$ then, $f_{\mathfrak{u}\mathfrak{v}}$ appears in $B^{\mathcal{O}}_{\mathfrak{t}\mathfrak{t}}$ with coefficient $p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})$ and, by induction, $\nu_x\big(p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) \geq \frac{1}{2}\big(\deg(\mathfrak{t},\mathfrak{u}) + \deg(\mathfrak{t},\mathfrak{v})\big)$. Therefore,

$$\begin{aligned}\nu_x\big(c_{\mathfrak{t}} p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) &= \nu_x(c_{\mathfrak{t}}) + \nu_x\big(p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) \geq \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t}) + \tfrac{1}{2}(\deg(\mathfrak{t},\mathfrak{u}) + \deg(\mathfrak{t},\mathfrak{v}))\\ &= \tfrac{1}{2}\big(\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{u}) + \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{v})\big).\end{aligned}$$

It follows that if $f_{\mathfrak{u}\mathfrak{v}}$ appears in $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ with non-zero coefficient $a_{\mathfrak{u}\mathfrak{v}}$ then $\nu_x(a_{\mathfrak{u}\mathfrak{v}}) \geq \frac{1}{2}\big(\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{u}) + \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{v})\big)$. If $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ now has the required properties then we can set $B_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$. Otherwise, let $(\mathfrak{s},\mathfrak{t})$ be a pair of tableau which is minimal with respect to dominance such that the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ is of the form $b_{\mathfrak{s}\mathfrak{t}} + c_{\mathfrak{s}\mathfrak{t}}$ with $c_{\mathfrak{s}\mathfrak{t}} \neq 0$, $\nu_x(c_{\mathfrak{s}\mathfrak{t}}) \geq 0$, $b_{\mathfrak{s}\mathfrak{t}} \in x^{-1}K[x^{-1}]$ and $\nu_x(b_{\mathfrak{s}\mathfrak{t}}) \geq \frac{1}{2}\big(\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s}) + \deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t})\big)$. Replacing $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ with $A_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} - c_{\mathfrak{s}\mathfrak{t}} B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}$ and continuing in this way we will, in a finite number of steps, construct an element $B'_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ with all of the required properties.

By the uniqueness statement in Theorem 6.2, $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B'_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ so this proves the proposition for the polynomials $p^{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}_{\mathfrak{u}\mathfrak{v}}(x^{-1})$.

Finally, suppose that $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\boldsymbol{\lambda})$ with $(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t}^{\boldsymbol{\lambda}}) \rhd (\mathfrak{s},\mathfrak{t})$. Without loss of generality, suppose that $\mathfrak{s} = \mathfrak{a}(r,r+1)$ where $\mathfrak{a} \in \mathrm{Std}(\mathbf{i})$, for $\mathbf{i} \in I^n$, and $\mathfrak{a} \rhd \mathfrak{s}$. Using Lemma 4.23,

$$\begin{aligned}\psi^{\mathcal{O}}_r B^{\mathcal{O}}_{\mathfrak{a}\mathfrak{t}} &= \sum_{(\mathfrak{u},\mathfrak{v})\ \underline{\blacktriangleright}\ (\mathfrak{a},\mathfrak{t})} p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\psi^{\mathcal{O}}_r f_{\mathfrak{u}\mathfrak{v}}\\ &= \sum_{(\mathfrak{u},\mathfrak{v})\ \underline{\blacktriangleright}\ (\mathfrak{a},\mathfrak{t})} p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\Big(\beta_r(\mathfrak{u})f_{\mathfrak{u}(r,r+1),\mathfrak{v}} - \delta_{i_r i_{r+1}}\frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{u})}}{[\rho_r(\mathfrak{u})]}f_{\mathfrak{u}\mathfrak{v}}\Big).\end{aligned}$$

By induction, $\nu_x(p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}) \ge \frac12(\deg(\mathfrak{a},\mathfrak{u}) + \deg(\mathfrak{t},\mathfrak{v}))$. Therefore, using Lemma 5.13 (as in the proof of Theorem 3.22), it follows that if $c_{\mathfrak{u}\mathfrak{v}} \ne 0$ is the coefficient of $f_{\mathfrak{u}\mathfrak{v}}$ in the last equation then $\nu_x(c_{\mathfrak{u}\mathfrak{v}}) \ge \frac12\big(\deg(\mathfrak{s},\mathfrak{u}) + \deg(\mathfrak{t},\mathfrak{v})\big)$. Hence, the proposition follows by repeating the argument of the last paragraph. □

6.2. **A distinguished homogeneous basis of $\mathcal{H}^\Lambda_n(K)$.** This section uses Theorem 6.2 to construct a new graded cellular basis of $\mathcal{H}^\Lambda_n(K)$. The existence of such a basis is not automatically guaranteed by Theorem 6.2 because the elements $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$, for $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}^\Lambda_n)$, are not necessarily homogeneous.

The isomorphisms $K \cong \mathcal{O}/x\mathcal{O} \cong \widehat{\mathcal{O}}/x\widehat{\mathcal{O}}$ extend to K-algebra isomorphisms

$$\mathcal{H}^\Lambda_n(K) \cong \mathcal{H}^\Lambda_n(\mathcal{O}) \otimes_{\mathcal{O}} K \cong \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} 1_K.$$

We identify these three K-algebras.

6.5. **Lemma.** *Suppose that $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}^\Lambda_n)$. Then*

$$B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K = \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} a_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}},$$

for some $a_{\mathfrak{u}\mathfrak{b}} \in K$. In particular, the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg\mathfrak{s} + \deg\mathfrak{t}$ is non-zero.

Proof. This is immediate from Theorem 6.2 (and Corollary 5.9). □

Recall from Step 2 in the proof of Theorem 6.2 that for each $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ there exists an element $D^{\mathcal{O}}_{\mathfrak{v}} \in \mathcal{H}^\Lambda_n(\mathcal{O})$ such that $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \equiv (D^{\mathcal{O}}_{\mathfrak{s}})^\star B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D^{\mathcal{O}}_{\mathfrak{t}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}$.

6.6. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$.*

a) *If $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ let $D_{\mathfrak{v}}$ be the homogeneous component of $D^{\mathcal{O}}_{\mathfrak{v}} \otimes 1_K$ of degree $\deg\mathfrak{v} - \deg\mathfrak{t}^{\boldsymbol{\lambda}}$.*
b) *Define $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ to be the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \otimes 1_K$ of degree $2\deg\mathfrak{t}^{\boldsymbol{\lambda}}$. More generally, if $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ define $B_{\mathfrak{s}\mathfrak{t}} = D^\star_{\mathfrak{s}} B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D_{\mathfrak{t}}$.*

By Theorem 6.2, $(B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda})^\star = B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ which implies that $B^\star_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$. Consequently, if $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $B^\star_{\mathfrak{s}\mathfrak{t}} = B_{\mathfrak{t}\mathfrak{s}}$. If $B_{\mathfrak{s}\mathfrak{t}} \ne 0$ then, by construction, $B_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg\mathfrak{s} + \deg\mathfrak{t}$. Unfortunately, it is not clear from the definitions that $B_{\mathfrak{s}\mathfrak{t}}$ is non-zero.

6.7. **Proposition.** *Suppose that $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}(\mathcal{P}^\Lambda_n)$. Then*

$$B_{\mathfrak{s}\mathfrak{t}} \equiv \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} b_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}} \qquad \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for some $b_{\mathfrak{u}\mathfrak{v}} \in K$. In particular, $B_{\mathfrak{s}\mathfrak{t}} \ne 0$.

Proof. Fix $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and suppose that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. If $\mathfrak{s} = \mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ then $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \otimes 1_K$ of degree $2\deg \mathfrak{t}^{\boldsymbol{\lambda}}$, so the result is just Lemma 6.5 in this case. Now consider the case when $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}$ is an arbitrary standard $\boldsymbol{\lambda}$-tableau. Then, since $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \equiv \psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$,

$$B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}} \otimes 1_K \equiv (\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \otimes 1_K)(D^{\mathcal{O}}_{\mathfrak{t}} \otimes 1_K) \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}.$$

Looking at the homogeneous component of degree $\deg \mathfrak{t}^{\boldsymbol{\lambda}} + \deg \mathfrak{t}$ shows that

$$B_{\mathfrak{t}^\lambda\mathfrak{t}} = B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D_{\mathfrak{t}} \equiv \psi_{\mathfrak{t}^\lambda\mathfrak{t}} + \sum_{\mathfrak{v} \blacktriangleright \mathfrak{t}} a_{\mathfrak{t}^\lambda\mathfrak{v}} \psi_{\mathfrak{t}^\lambda\mathfrak{v}} \qquad \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}},$$

by Lemma 6.5. Set $b_{\mathfrak{t}^\lambda\mathfrak{v}} = a_{\mathfrak{t}^\lambda\mathfrak{v}}$ with $b_{\mathfrak{t}^\lambda\mathfrak{t}} = 1$. Similarly,

$$D^\star_{\mathfrak{s}} \psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \equiv D^\star_{\mathfrak{s}} B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B_{\mathfrak{s}\mathfrak{t}^\lambda} \equiv \sum_{\mathfrak{u} \,\underline{\blacktriangleright}\, \mathfrak{s}} b_{\mathfrak{u}\mathfrak{t}^\lambda} \psi_{\mathfrak{u}\mathfrak{t}^\lambda} \qquad \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}},$$

where $b_{\mathfrak{u}\mathfrak{t}^\lambda} = a_{\mathfrak{t}^\lambda\mathfrak{u}}$ with $b_{\mathfrak{s}\mathfrak{t}^\lambda} = 1$. By Corollary 5.9, $\{\psi_{\mathfrak{u}\mathfrak{v}}\}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda(K)$ so, working modulo $\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$,

$$\begin{aligned} B_{\mathfrak{s}\mathfrak{t}} = D^\star_{\mathfrak{s}} B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D_{\mathfrak{t}} &\equiv \sum_{\mathfrak{v} \,\underline{\blacktriangleright}\, \mathfrak{t}} b_{\mathfrak{t}^\lambda\mathfrak{v}} D^\star_{\mathfrak{s}} \psi_{\mathfrak{t}^\lambda\mathfrak{v}} \equiv \sum_{\mathfrak{v} \,\underline{\blacktriangleright}\, \mathfrak{t}} \sum_{\mathfrak{u} \,\underline{\blacktriangleright}\, \mathfrak{s}} b_{\mathfrak{t}^\lambda\mathfrak{v}} b_{\mathfrak{u}\mathfrak{t}^\lambda} \psi_{\mathfrak{u}\mathfrak{v}} \\ &= \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})} b_{\mathfrak{u}\mathfrak{t}^\lambda} b_{\mathfrak{t}^\lambda\mathfrak{v}} \psi_{\mathfrak{u}\mathfrak{v}} \qquad \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}. \end{aligned}$$

Setting $b_{\mathfrak{u}\mathfrak{v}} = \mathfrak{b}_{\mathfrak{s}\mathfrak{t}^\lambda} b_{\mathfrak{t}^\lambda\mathfrak{v}}$ completes the proof. $\square$

6.8. *Remark.* If K is a field of characteristic zero then it follows from Proposition 6.4 that $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ is a linear combinations of homogeneous components of degree greater than or equal to $\deg \mathfrak{s} + \deg \mathfrak{t}$. As a consequence, $B_{\mathfrak{s}\mathfrak{t}}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$. As far as we can see, if K is a field of positive characteristic then it is not true in general that $B_{\mathfrak{s}\mathfrak{t}}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$.

We can now prove Theorem B from the introduction.

6.9. **Theorem.** *Suppose that K is a field. Then $\{ B_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda(K)$ with cellular algebra automorphism $\star$.*

Proof. By Proposition 6.7 and Corollary 5.9, $\{ B_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \}$ is a basis of $\mathcal{H}_n^\Lambda(K)$. By definition, if $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda)$ then $B_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$ and $B^\star_{\mathfrak{s}\mathfrak{t}} = B_{\mathfrak{t}\mathfrak{s}}$. Therefore, the basis $\{B_{\mathfrak{s}\mathfrak{t}}\}$ satisfies (GC$_1$), (GC$_3$) and (GC$_d$) from Definition 2.4. Finally, since $B_{\mathfrak{s}\mathfrak{t}} \equiv D^\star_{\mathfrak{s}} B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D_{\mathfrak{t}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, (GC$_2$) follows by repeating the argument from Step 3 in the proof of Theorem 6.2. $\square$

The graded cellular basis $\{ B_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \}$ of $\mathcal{H}_n^\Lambda(K)$ is distinguished in the sense that, unlike $\psi_{\mathfrak{s}\mathfrak{t}}$, the element $B_{\mathfrak{s}\mathfrak{t}}$ depends only on $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda)$ and not on a choice of reduced expressions for the permutations $d(\mathfrak{s})$ and $d(\mathfrak{t})$.

Appendix A. Seminormal forms for the linear quiver

In this appendix we show how the results in this paper work when $e = 0$ so that $\xi \in K$ is either not a root of unity or $\xi = 1$ and K is a field of characteristic zero. Interestingly, all of the results in this appendix apply equally well when $e = 0$ or when $e \geq n$. The main difference is that in order to define a modular system we have to leave the case where the cyclotomic parameters $Q_1, \dots, Q_\ell$ are *integral*, that is, when $Q_l = [\kappa_l]$ for $1 \leq l \leq \ell$. This causes quite a few notational inconveniences, but otherwise the story is much the same as for the case when $e > 0$. We do not

develop the full theory of "0-idempotent subrings" here. Rather, we show just one way of proving the results in this paper when $e = 0$.

Fix a field K and $0 \neq \xi \in K$ of quantum characteristic e. That is, either $\xi = 1$ and K is a field of characteristic zero or $\xi^d \neq 1$ for $d \in \mathbb{Z}$. The multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ is arbitrary.

Let $\mathcal{O} = \mathbb{Z}[x, \xi]_{(x)}$ be the localisation of $\mathbb{Z}[x, \xi]$ at the principal ideal generated by x. Let $\mathscr{K} = \mathbb{Q}(x, \xi)$ be the field of fractions of $\mathcal{O}$. Define $\mathcal{H}_n^\Lambda(\mathcal{O})$ to be the cyclotomic Hecke algebra of type A with Hecke parameter $t = \xi$, a unit in $\mathcal{O}$, and cyclotomic parameters

$$Q_l = x^l + [\kappa_l], \qquad \text{for } 1 \leq l \leq \ell,$$

where, as before, $[k] = [k]_t$ for $k \in \mathbb{Z}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$ is split semisimple in view of Ariki's semisimplicity condition [1]. Moreover, by definition, $\mathcal{H}_n^\Lambda(K) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$, where we consider K as an $\mathcal{O}$-module by setting x act on K as multiplication by zero.

Define a new **content** function for $\mathcal{H}_n^\Lambda(\mathcal{O})$ by setting

$$C_\gamma = t^{c-r} x^l + [\kappa_l + c - r],$$

for a node $\gamma = (l, r, c)$. We will also need the previous definition of contents below. If $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ is a tableau and $1 \leq k \leq n$ then set $C_k(\mathfrak{t}) = C_\gamma$, where γ is the unique node such that $\mathfrak{t}(\gamma) = k$.

As in Section 2.5, let $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ be the Murphy basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$. Then the analogue of Lemma 2.9 is that if $1 \leq r \leq n$ then

$$m_{\mathfrak{s}\mathfrak{t}} L_r = C_r(\mathfrak{t}) m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathcal{O}$. As in Section 3.1 define a $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ to be a basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of simultaneous two-sided eigenvectors for $L_1, \dots, L_n$ such that $f_{\mathfrak{s}\mathfrak{t}}^* = f_{\mathfrak{t}\mathfrak{s}}$.

Define a **seminormal coefficient system** for $\mathcal{H}_n^\Lambda(\mathcal{O})$ to be a set of scalars $\boldsymbol{\alpha} = \{\alpha_r(\mathfrak{s})\}$ which satisfies (3.9) and such that if $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathfrak{u} = \mathfrak{s}(r, r+1) \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ then

$$\alpha_r(\mathfrak{s})\alpha_r(\mathfrak{u}) = \frac{(1 - C_r(\mathfrak{s}) + tC_r(\mathfrak{u}))(1 + tC_r(\mathfrak{s}) - C_r(\mathfrak{u}))}{P_r(\mathfrak{s})P_r(\mathfrak{u})}, \tag{A1}$$

where $P_r(\mathfrak{s}) = C_r(\mathfrak{u}) - C_r(\mathfrak{s})$, and where $\alpha_r(\mathfrak{s}) = 0$ if $\mathfrak{u} \notin \operatorname{Std}(\mathcal{P}_n^\Lambda)$.

As in Theorem 3.14, each seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is determined by a seminormal coefficient system $\boldsymbol{\alpha} = \{\alpha_r(\mathfrak{s})\}$, such that

$$T_r f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} + \frac{1 + (t-1)C_{r+1}(\mathfrak{s})}{P_r(\mathfrak{s})} f_{\mathfrak{s}\mathfrak{t}}, \qquad \text{where } \mathfrak{u} = \mathfrak{s}(r, r+1),$$

together with a set of scalars $\{\, \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$. Notice that $I = \mathbb{Z}$, since $e = 0$, so if $\mathbf{i} \in I^n$ then $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$ if and only if $c_r(\mathfrak{t}) = i_r$ and, in turn, this is equivalent to the constant term of $C_r(\mathfrak{t})$ being equal to $[i_r]$, for $1 \leq r \leq n$. Arguing as in Lemma 4.4,

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}} \in \mathcal{H}_n^\Lambda(\mathcal{O}).$$

With these definitions in place all of the arguments in Chapter 4 go through with only minor changes. In particular, if $1 \leq r \leq n$ and $\mathbf{i} \in I^n$ then Definition 4.14 should be replaced by

$$\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (T_r + 1)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1} \\ (T_r L_r - L_r T_r) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1} + 1, \\ (T_r L_r - L_r T_r)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

and $y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \big(L_r - C_r(\mathfrak{t})\big) f_{\mathbf{i}}^{\mathcal{O}}$ where, as before, $M_r = 1 - L_r + tL_{r+1}$. With these new definitions, if $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$, for $\mathbf{i} \in I^m$, and $1 \le r \le n$ then Lemma 4.23 becomes

$$\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = B_r(\mathfrak{s}) f_{\mathfrak{s}\mathfrak{t}} + \frac{\delta_{i_r i_{r+1}}}{P_r(\mathfrak{s})} f_{\mathfrak{u}\mathfrak{t}},$$

where $\mathfrak{u} = \mathfrak{s}(r, r+1)$ and

$$B_r(\mathfrak{s}) = \begin{cases} \frac{\alpha_r(\mathfrak{s})}{1 - C_r(\mathfrak{s}) + tC_{r+1}(\mathfrak{s})}, & \text{if } i_r = i_{r+1},\\ \alpha_r(\mathfrak{s}) P_r(\mathfrak{s}), & \text{if } i_r = i_{r+1} + 1,\\ \frac{\alpha_r(s) P_r(\mathfrak{s})}{1 - C_r(\mathfrak{s}) + tC_{r+1}(\mathfrak{s})}, & \text{otherwise.} \end{cases}$$

Observe that if $\mathfrak{u} = \mathfrak{s}(r, r+1)$ is a standard tableau then, using (A1), the definitions imply that

$$B_r(\mathfrak{s}) B_r(\mathfrak{u}) = \begin{cases} \frac{1}{P_r(\mathfrak{s}) P_r(\mathfrak{u})}, & \text{if } i_r = i_{r+1},\\ (1 - C_r(\mathfrak{s}) + tC_r(\mathfrak{u}))(1 + tC_r(\mathfrak{s}) - C_r(\mathfrak{u})), & \text{if } i_r \leftrightarrows i_{r+1},\\ (1 + tC_r(\mathfrak{s}) - C_r(\mathfrak{u})), & \text{if } i_r \rightarrow i_{r+1},\\ (1 - C_r(\mathfrak{s}) + tC_r(\mathfrak{u})), & \text{if } i_r \leftarrow i_{r+1},\\ 1, & \text{otherwise.} \end{cases}$$

Comparing this with Lemma 4.26, it is now easy to see that analogues of Proposition 4.28 and Proposition 4.29 both hold in this situation. Hence, repeating the arguments of Section 4.4, Theorem A also holds. Similarly, the construction of the bases in Chapter 5 and Chapter 6 now goes though largely without change.

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References

[1] S. Ariki, *On the semi–simplicity of the Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, J. Algebra, **169** (1994), 216–225. [Pages 13, 46.]

[2] ———, *On the classification of simple modules for cyclotomic Hecke algebras of type* $G(m,1,n)$ *and Kleshchev multipartitions*, Osaka J. Math., **38** (2001), 827–837. [Page 10.]

[3] S. Ariki and K. Koike, *A Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math., **106** (1994), 216–243. [Pages 6, 13.]

[4] S. Ariki, A. Mathas, and H. Rui, *Cyclotomic Nazarov–Wenzl algebras*, Nagoya J. Math., **182** (2006), 47–134. (Special issue in honour of George Lusztig). [Pages 4, 10, 13, 18.]

[5] J. Brundan and A. Kleshchev, *Schur-Weyl duality for higher levels*, Selecta Math., **14** (2008), 1–57. [Page 6.]

[6] ———, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484. [Pages 3, 5, 11, 27, 34, 35.]

[7] ———, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942. [Pages 4, 21.]

[8] J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. Reine Angew. Math., **655** (2011), 61–87. arXiv:math.RT/0901.0218. [Pages 3, 9, 21, 36, 39, 41.]

[9] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra III: category* $\mathscr{O}$, Represent. Theory, **15** (2011), 170–243. arXiv:0812.1090. [Page 21.]

[10] R. Dipper, G. James, and A. Mathas, *Cyclotomic q–Schur algebras*, Math. Z., **229** (1999), 385–416. [Page 10.]

[11] R. Dipper and A. Mathas, *Morita equivalences of Ariki–Koike algebras*, Math. Zeit., **240** (2002), 579–610. [Pages 6, 12.]

[12] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., **123** (1996), 1–34. [Pages 7, 10.]

[13] A. E. Hoffnung and A. D. Lauda, *Nilpotency in type A cyclotomic quotients*, J. Algebraic Combin., **32** (2010), 533–555. [Page 35.]

[14] J. Hu and A. Mathas, *Cyclotomic quiver Schur algebras I: linear quivers*, preprint 2011. arXiv:1110.1699. [Page 21.]

[15] ———, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642. [Pages 3, 5, 7, 8, 12, 21, 36, 37, 38, 39, 41.]
[16] G. D. James and A. Mathas, *A q–analogue of the Jantzen–Schaper Theorem*, Proc. Lond. Math. Soc. (3), **74** (1997), 241–274. [Pages 4, 18.]
[17] ———, *The Jantzen sum formula for cyclotomic q–Schur algebras*, Trans. Amer. Math. Soc., **352** (2000), 5381–5404. [Pages 4, 10, 13, 18, 19.]
[18] G. D. James and G. E. Murphy, *The determinant of the Gram matrix for a Specht module*, J. Algebra, **59** (1979), 222–235. [Pages 4, 18.]
[19] V. G. Kac, *Infinite dimensional Lie algebras*, CUP, Cambridge, third ed., 1994. [Page 5.]
[20] M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc., **363** (2008), 2685–2700. arXiv:0804.2080. [Page 1.]
[21] ———, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory, **13** (2009), 309–347. [Page 1.]
[22] A. Kleshchev, A. Mathas, and A. Ram, *Universal Specht modules for cyclotomic Hecke algebras*, Proc. Lond. Math. Soc., **105** (2012), 1245–1289. arXiv:1102.3519. [Pages 11, 36, 39.]
[23] G. Li, *Integral Basis Theorem of cyclotomic Khovanov-Lauda-Rouquier Algebras of type A*, PhD thesis, University of Sydney, 2012. [Page 3.]
[24] A. Mathas, *Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, **15**, Amer. Math. Soc., 1999. [Page 9.]
[25] ———, *Matrix units and generic degrees for the Ariki-Koike algebras*, J. Algebra, **281** (2004), 695–730. [Pages 18, 22.]
[26] ———, *Seminormal forms and Gram determinants for cellular algebras*, J. Reine Angew. Math., **619** (2008), 141–173. With an appendix by M. Soriano. [Pages 13, 17, 22, 23.]
[27] G. E. Murphy, *The idempotents of the symmetric group and Nakayama's conjecture*, J. Algebra, **81** (1983), 258–265. [Pages 3, 13, 22, 23.]
[28] A. Y. Okounkov and A. M. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math., **2** (1996), 581–605. [Page 13.]
[29] R. Rouquier, *2-Kac-Moody algebras*, preprint 2008. arXiv:0812.5023. [Page 1.]
[30] S. Ryom-Hansen, *The Schaper formula and the Lascoux, Leclerc and Thibon algorithm*, Lett. Math. Phys., **64** (2003), 213–219. [Page 21.]
[31] ———, *Young's seminormal form and simple modules for S_n in characteristic p*, preprint, (2011). [Page 25.]
[32] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, **67**, Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. [Page 41.]
[33] C. Stroppel and B. Webster, *Quiver Schur algebras and q-Fock space*, preprint 2011. arXiv:1110.1115. [Page 27.]
[34] A. Young, *On Quantitative Substitutional Analysis I*, Proc. Lond. Math. Soc. (1), **33** (1900), 97–145. [Pages 4, 14.]
[35] X. Yvonne, *A conjecture for q-decomposition matrices of cyclotomic v-Schur algebras*, J. Algebra, **304** (2006), 419–456. [Page 21.]

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